

Solutions to In-Class Problems Week 10, Fri.

Problem 1.

A certain company wants to have security for their computer systems. So they have given everyone a name and password. A length 10 word containing each of the characters:

a, d, e, f, i, l, o, p, r, s,

is called a *cword*. A password will be a cword which does not contain any of the subwords "fails", "failed", or "drop".

For example, the following two words are passwords:

adefiloprs, srpolifeda,

but the following three cwords are not:

adropeflis, failedrops, dropefails.

(a) How many cwords contain the subword "drop"?

Solution. Such cwords are obtainable by taking the word "drop" and the remaining 6 letters in any order. There are $7!$ permutations of these 7 items. ■

(b) How many cwords contain both "drop" and "fails"?

Solution. Take the words "drop" and "fails" and the remaining letter "e" in any order. So there are $3!$ such cwords. ■

(c) Use the Inclusion-Exclusion Principle to find a simple formula for the number of passwords.

Solution. There are $7!$ cwords that contain "drop", $6!$ that contain "fails", and $5!$ that contain "failed". There are $3!$ cwords containing both "drop" and "fails". No cword can contain both "fails" and "failed". The cwords containing both "drop" and "failed" come from taking the subword "failedrop" and the remaining letter "s" in any order, so there are $2!$ of them. So by Inclusion-exclusion, we have the number of cwords containing at least one of the three forbidden subwords is

$$(7! + 6! + 5!) - (3! + 0 + 2!) + 0 = 5!(49) - 8.$$

Among the $10!$ cwords, the remaining ones are passwords, so the number of passwords is

$$10! - 7! - 6! - 5! + 3! + 2! = 3,622,928.$$

Problem 2.

Solve the following counting problems by defining an appropriate mapping (bijective or k -to-1) between a set whose size you know and the set in question.

- (a) How many different ways are there to select a dozen donuts if four varieties are available?

Solution. There is a bijection from selections of a dozen donuts to 15-bit sequences with exactly 3 ones. In particular, suppose that the varieties are glazed, chocolate, lemon, and Boston creme. Then a selection of g glazed, c chocolate, l lemon, and b Boston creme maps to the sequence:

$$(g \text{ 0's}) 1 (c \text{ 0's}) 1 (l \text{ 0's}) 1 (b \text{ 0's})$$

Therefore, the number of selections is equal to the number of 15-bit sequences with exactly 3 ones, which is:

$$\frac{15!}{3! 12!} = \binom{15}{3}$$

■

- (b) In how many ways can Mr. and Mrs. Grumperson distribute 13 identical pieces of coal to their two —no, three! —children for Christmas?

Solution. There is a bijection from 15-bit strings with two ones. In particular, the bit string $0^a 1 0^b 1 0^c$ maps to the assignment of a coals to the first child, b coals to the second, and c coals to the third. Therefore, there are $\binom{15}{2}$ assignments. ■

- (c) How many solutions over the nonnegative integers are there to the inequality:

$$x_1 + x_2 + \dots + x_{10} \leq 100$$

Solution. There is a bijection from 110-bit sequences with 10 ones to solutions to this equation. In particular, x_i is the number of zeros before the i -th one but after the $(i-1)$ -st one (or the beginning of the sequence). Therefore, there are $\binom{110}{10}$ solutions. ■

(d) We want to count step-by-step paths between points in the plane with integer coordinates. Only two kinds of step are allowed: a right-step which increments the x coordinate, and an up-step which increments the y coordinate.

- (i) How many paths are there from $(0, 0)$ to $(20, 30)$?

Solution. $\binom{50}{20}$.

There is a bijection from 50-bit sequences with 20 zeros and 30 ones. The sequence (b_1, \dots, b_{30}) maps to a path where the i -th step is right if $b_i = 0$ and up if $b_i = 1$. Therefore, the number of paths is equal to $\binom{50}{20}$. ■

- (ii) How many paths are there from $(0, 0)$ to $(20, 30)$ that go through the point $(10, 10)$?

Solution. $\binom{20}{10} \cdot \binom{30}{10}$.

There is a bijection between the paths from $(20, 30)$ that go through $(10, 10)$ and set of pairs of paths consisting of path from $(0, 0)$ to $(10, 10)$ and a path from $(10, 10)$ to $(20, 30)$. So the number of paths through $(10, 10)$ is the product of the sizes of these two sets of paths. ■

(iii) How many paths are there from $(0, 0)$ to $(20, 30)$ that do *not* go through either of the points $(10, 10)$ and $(15, 20)$?

Hint: Let P be the set of paths from $(0, 0)$ to $(20, 30)$, N_1 be the paths in P that go through $(10, 10)$ and N_2 be the paths in P that go through $(15, 20)$.

Solution.

$$\binom{50}{20} - \binom{20}{10} \cdot \binom{30}{10} - \binom{30}{15} \cdot \binom{15}{5} + \binom{20}{10} \cdot \binom{15}{5} \cdot \binom{15}{5}.$$

$N_1 \cap N_2$ is the set of paths from $(0, 0)$ to $(20, 30)$ that go through both $(10, 10)$ and $(15, 20)$. So $P - (N_1 \cup N_2)$ is the set of paths to be counted. Now we have

$$\begin{aligned} |P - (N_1 \cup N_2)| &= |P| - |N_1 \cup N_2| \\ &= |P| - |N_1| - |N_2| + |N_1 \cap N_2| \quad \text{by Inclusion-Exclusion.} \end{aligned}$$

Part (ii) shows how to calculate $|N_i|$. Also, there is a bijection between $N_1 \cap N_2$ and the set of triples consisting of a path $(0, 0)$ to $(10, 10)$, a path from $(10, 10)$ to $(15, 20)$, and a path from $(15, 20)$ to $(20, 30)$. So the size of $N_1 \cap N_2$ is the product of the sizes of these three sets of paths. ■

Problem 3.

Here are the solutions to the next 10 problem parts, in no particular order.

$$n^m \quad m^n \quad \frac{n!}{(n-m)!} \quad \binom{n+m}{m} \quad \binom{n-1+m}{m} \quad \binom{n-1+m}{n} \quad 2^{mn}$$

(a) How many solutions over the natural numbers are there to the inequality $x_1 + x_2 + \cdots + x_n \leq m$? _____

Solution.

$$\binom{n+m}{m}$$

This is the same as the number of solutions to the equation the equality $x_1 + x_2 + \cdots + x_n + y = m$, and which has a bijection to sequences with m stars and n bars. ■

(b) How many length m words can be formed from an n -letter alphabet, if no letter is used more than once? _____

Solution.

$$\frac{n!}{(n-m)!}$$

There are n choices for the first letter, $n - 1$ choices for the second letter, \dots $n - m + 1$ choices for the m th letter, so by the Generalized Product rule, the number of words is

$$n \cdot (n - 1) \cdots (n - m + 1).$$

■

- (c) How many length m words can be formed from an n -letter alphabet, if letters can be reused? _____

Solution. n^m by the Product Rule.

■

- (d) How many binary relations are there from set A to set B when $|A| = m$ and $|B| = n$? _____

Solution.

$$2^{mn}$$

The graph of a binary relations from A to B is a subset of $A \times B$. There are on 2^{mn} such subsets because $|A \times B| = mn$. ■

- (e) How many injections are there from set A to set B , where $|A| = m$ and $|B| = n \geq m$? _____

Solution.

$$\frac{n!}{(n-m)!}$$

There is a bijection between the injections and the length m sequences of distinct elements of B . By the Generalized Product rule, the number of such sequences is

$$n \cdot (n - 1) \cdots (n - m + 1).$$

■

- (f) How many ways are there to place a total of m distinguishable balls into n distinguishable urns, with some urns possibly empty or with several balls? _____

Solution.

$$n^m$$

There is a bijection between a placement of the balls and length m sequence whose i th element is the urn where the i th ball is placed. So the number of placements is the same as the number of length m sequences of elements from a size- n set. ■

- (g) How many ways are there to place a total of m indistinguishable balls into n distinguishable urns, with some urns possibly empty or with several balls? _____

Solution.

$$\binom{n-1+m}{m}$$

This is the same as the number of selections of m donuts with n possible flavors, which is the number of sequences with m stars and $n-1$ bars. ■

- (h) How many ways are there to put a total of m distinguishable balls into n distinguishable urns with at most one ball in each urn? _____

Solution.

$$\frac{n!}{(n-m)!}$$

There is a bijection between a placement of balls and a length m sequence whose i th element is the urn containing the i th ball. So the number of ball placements is the same as number of length m sequences of distinct elements from a set of n elements. ■

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