


Mathematics for Computer Science
 MIT 6.042J/18.062J


Inclusion-exclusion Counting practice


 Albert R Meyer, April 16, 2010 lec 10F.1

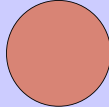

Sum Rule

$$|A \cup B| = |A| + |B|$$


A




B



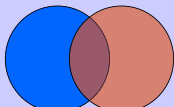
for disjoint sets A, B


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Sum Rule


$$|A \cup B| = ?$$


A



B

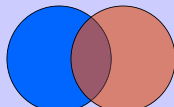
What if **not** disjoint?


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Inclusion-Exclusion


$$|A \cup B| = |A| + |B| - |A \cap B|$$


A



B

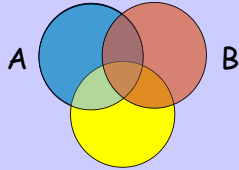
What if **not** disjoint?


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Inclusion-Exclusion (3 Sets)


$$\begin{aligned}
 |A \cup B \cup C| = & \\
 & |A| + |B| + |C| \\
 & - |A \cap B| - |A \cap C| - |B \cap C| \\
 & + |A \cap B \cap C|
 \end{aligned}$$


A



B

C



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Incl-Excl Formula: Proof

A town has n clubs.

Each club S_i has a *secretary* M_i who knows if person x is a club member:

$$\begin{aligned}
 M_i(x) &= 1 \text{ if } x \text{ in } S_i, \\
 &= 0 \text{ if } x \text{ not in } S_i.
 \end{aligned}$$


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Incl-Excl Formula: Proof

So $|A| = \sum_{x \in \text{people}} M_A(x)$
 $M_i(x)M_j(x)$ is sec'y for $S_i \cap S_j$, so
 $|S_i \cap S_j| = \sum_x M_i(x) \cdot M_j(x)$
 $|S_i \cap S_j \cap S_k| = \sum_x M_i(x)M_j(x)M_k(x)$
 etc.

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Incl-Excl Formula: Proof

Let $D ::= S_1 \cup S_2 \cup \dots \cup S_n$
 sec'y $M_D(x)=0$ iff $M_i(x)=0$ for
 all n clubs. So
 $1-M_D(x) =$
 $(1-M_1(x))(1-M_2(x)) \dots (1-M_n(x))$

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Incl-Excl Formula: Proof

$1-M_D(x) = (1-M_1(x))(1-M_2(x)) \dots (1-M_n(x))$
 so...
 $M_D(x) = \sum_i M_i(x)$
 $- \sum_{i < j} M_i(x)M_j(x)$
 $+ \sum_{i < j < k} M_i(x)M_j(x)M_k(x)$
 \vdots
 $+ (-1)^{n+1} M_1(x)M_2(x) \dots M_n(x)$

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Incl-Excl Formula: Proof

$1-M_D(x) = (1-M_1(x))(1-M_2(x)) \dots (1-M_n(x))$
 now sum both sides over x
 $M_D(x) = \sum_i M_i(x)$
 $- \sum_{i < j} M_i(x)M_j(x)$
 $+ \sum_{i < j < k} M_i(x)M_j(x)M_k(x)$
 \vdots
 $+ (-1)^{n+1} M_1(x)M_2(x) \dots M_n(x)$

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Incl-Excl Formula: Proof

$1-M_D(x) = (1-M_1(x))(1-M_2(x)) \dots (1-M_n(x))$
 now sum both sides over x
 $|D| = \sum_i |S_i|$
 $- \sum_{i < j} M_i(x)M_j(x)$
 $+ \sum_{i < j < k} M_i(x)M_j(x)M_k(x)$
 \vdots
 $+ (-1)^{n+1} M_1(x)M_2(x) \dots M_n(x)$

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Incl-Excl Formula: Proof

$1-M_D(x) = (1-M_1(x))(1-M_2(x)) \dots (1-M_n(x))$
 now sum both sides over x
 $|D| = \sum_i |S_i|$
 $- \sum_{i < j} |S_i \cap S_j|$
 $+ \sum_{i < j < k} M_i(x)M_j(x)M_k(x)$
 \vdots
 $+ (-1)^{n+1} M_1(x)M_2(x) \dots M_n(x)$

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Incl-Excl Formula: Proof

$$1 - M_D(x) = (1 - M_1(x))(1 - M_2(x)) \cdots (1 - M_n(x))$$

now sum both sides over x

$$\begin{aligned}
 |D| &= \sum_i |S_i| \\
 &\quad - \sum_{i < j} |S_i \cap S_j| \\
 &\quad + \sum_{i < j < k} |S_i \cap S_j \cap S_k| \\
 &\quad \vdots \\
 &\quad + (-1)^{n+1} |M_1(x) M_2(x) \cdots M_n(x)|
 \end{aligned}$$



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Incl-Excl Formula: Proof

$$1 - M_D(x) = (1 - M_1(x))(1 - M_2(x)) \cdots (1 - M_n(x))$$

now sum both sides over x

$$\begin{aligned}
 |D| &= \sum_i |S_i| \\
 &\quad - \sum_{i < j} |S_i \cap S_j| \\
 &\quad + \sum_{i < j < k} |S_i \cap S_j \cap S_k| \\
 &\quad \vdots \\
 &\quad + (-1)^{n+1} |S_1 \cap S_2 \cdots \cap S_n|
 \end{aligned}$$



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Incl-Excl Formula: Proof

$$\begin{aligned}
 |D| &= \sum_i |S_i| \\
 &\quad - \sum_{i < j} |S_i \cap S_j| \\
 &\quad + \sum_{i < j < k} |S_i \cap S_j \cap S_k| \\
 &\quad \vdots \\
 &\quad + (-1)^{n+1} |S_1 \cap S_2 \cdots \cap S_n|
 \end{aligned}$$



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Team Problems

Problems 1–3



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Spring 2010

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