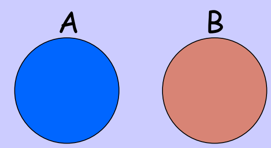
Mathematics for Computer Science

MIT 6.042J/18.062J

Inclusion-exclusion Counting practice

Sum Rule

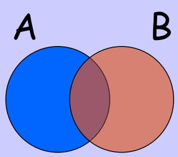
$$|A \cup B| = |A| + |B|$$



for disjoint sets A, B

Sum Rule

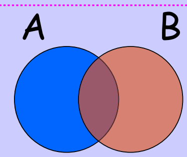
$$|A \cup B| = ?$$



What if not disjoint?

Inclusion-Exclusion

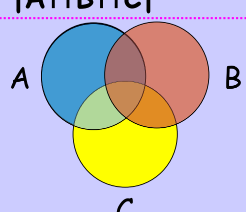
$$|A \cup B| = |A| + |B| - |A \cap B|$$



What if not disjoint?

Inclusion-Exclusion (3 Sets)

$$|A \cup B \cup C| = |A|+|B|+|C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



Incl-Excl Formula: Proof

A town has n clubs.
Each club S_i has a *secretary* M_i who knows if person x is a club member:

$M_i(x) = 1$ if x in S_i,
$\quad\quad = 0$ if x not in S_i.

Incl-Excl Formula: Proof

So $|A| = \sum_{x \in \text{people}} M_A(x)$

$M_i(x)M_j(x)$ is sec'y for $S_i \cap S_j$, so

$$|S_i \cap S_j| = \sum_x M_i(x) \cdot M_j(x)$$

$$|S_i \cap S_j \cap S_k| = \sum_x M_i(x) M_j(x) M_k(x)$$

etc.

Incl-Excl Formula: Proof

Let $D ::= S_1 \cup S_2 \cup \cdots \cup S_n$
sec'y $M_D(x)=0$ iff $M_i(x)=0$ for all n clubs. So
$1-M_D(x) =$

$$(1-M_1(x))(1-M_2(x))\cdots(1-M_n(x))$$

Incl-Excl Formula: Proof

$1-M_D(x) = (1-M_1(x))(1-M_2(x))\cdots(1-M_n(x))$
so...

$$\begin{aligned} M_D(x) = & \sum_i M_i(x) \\ & - \sum_{i<j} M_i(x)M_j(x) \\ & + \sum_{i<j<k} M_i(x)M_j(x)M_k(x) \\ & \vdots \\ & + (-1)^{n+1} M_1(x)M_2(x)\cdots M_n(x) \end{aligned}$$

Incl-Excl Formula: Proof

$1-M_D(x) = (1-M_1(x))(1-M_2(x))\cdots(1-M_n(x))$
now sum both sides over x

$$\begin{aligned} M_D(x) = & \sum_i M_i(x) \\ & - \sum_{i<j} M_i(x)M_j(x) \\ & + \sum_{i<j<k} M_i(x)M_j(x)M_k(x) \\ & \vdots \\ & + (-1)^{n+1} M_1(x)M_2(x)\cdots M_n(x) \end{aligned}$$

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$$\begin{aligned} |D| = & \sum_i |S_i| \\ & - \sum_{i<j} M_i(x)M_j(x) \\ & + \sum_{i<j<k} M_i(x)M_j(x)M_k(x) \\ & \vdots \\ & + (-1)^{n+1} M_1(x)M_2(x)\cdots M_n(x) \end{aligned}$$

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Incl-Excl Formula: Proof

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$$|D| = \sum_i |S_i| - \sum_{i<j} |S_i \cap S_j| + \sum_{i<j<k} |S_i \cap S_j \cap S_k| \; \vdots \; + (-1)^{n+1} M_1(x)M_2(x)\cdots M_n(x)$$

Incl-Excl Formula: Proof

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Incl-Excl Formula: Proof

$$|D| = \sum_i |S_i| - \sum_{i<j} |S_i \cap S_j| + \sum_{i<j<k} |S_i \cap S_j \cap S_k| \; \vdots \; + (-1)^{n+1} |S_1 \cap S_2 \cdots \cap S_n|$$

Team Problems

Problems

1—3