Solutions to In-Class Problems Week 10, Wed.

Problem 1.
The Tao of BOOKKEEPER: we seek enlightenment through contemplation of the word $BOOKKEEPER$.

(a) In how many ways can you arrange the letters in the word $POKE$?

Solution. There are $4!$ arrangements corresponding to the $4!$ permutations of the set $\{P, O, K, E\}$. ■

(b) In how many ways can you arrange the letters in the word BO_1O_2K? Observe that we have subscripted the O's to make them distinct symbols.

Solution. There are $4!$ arrangements corresponding to the $4!$ permutations of the set $\{B, O_1, O_2, K\}$. ■

(c) Suppose we map arrangements of the letters in BO_1O_2K to arrangements of the letters in $BOOK$ by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

O_2BO_1K	
KO_2BO_1	$BOOK$
O_1BO_2K	$OBOK$
KO_1BO_2	$KOBO$
BO_1O_2K	$\ldots$
BO_2O_1K	
$\ldots$	

(d) What kind of mapping is this, young grasshopper?

Solution. 2-to-1 ■

(e) In light of the Division Rule, how many arrangements are there of $BOOK$?

Solution. $4!/2$ ■

(f) Very good, young master! How many arrangements are there of the letters in $KE_1E_2PE_3R$?

Solution. $6!$ ■

(g) Suppose we map each arrangement of $KE_1E_2PE_3R$ to an arrangement of $KEEPER$ by erasing subscripts. List all the different arrangements of $KE_1E_2PE_3R$ that are mapped to $REPEEK$ in this way.

Solution. $RE_1PE_2E_3K$, $RE_1PE_3E_2K$, $RE_2PE_1E_3K$, $RE_2PE_3E_1K$, $RE_3PE_1E_2K$, $RE_3PE_2E_1K$ ■

(h) What kind of mapping is this?

Solution. 3!-to-1 ■

(i) So how many arrangements are there of the letters in $KEEPER$?

Solution. $6!/3!$ ■

(j) *Now you are ready to face the BOOKKEEPER!*
How many arrangements of $BO_1O_2K_1K_2E_1E_2PE_3R$ are there?

Solution. $10!$ ■

(k) How many arrangements of $BOOK_1K_2E_1E_2PE_3R$ are there?

Solution. $10!/2!$ ■

(l) How many arrangements of $BOOKKE_1E_2PE_3R$ are there?

Solution. $10!/(2! \cdot 2!)$ ■

(m) How many arrangements of $BOOKKEEPER$ are there?

Solution.

$$\binom{10}{1,2,2,3,1,1} ::= \frac{10!}{1!\ 2!\ 2!\ 3!\ 1!\ 1!} = \frac{10!}{(2!)^2\ 3!}$$

■

Remember well what you have learned: subscripts on, subscripts off.

This is the Tao of Bookkeeper.

(n) How many arrangements of $VOODOODOLL$ are there?

Solution.

$$\binom{10}{1,2,5,2} ::= \frac{10!}{1!\ 2!\ 5!\ 2!}$$

■

(o) How many length 52 sequences of digits contain exactly 17 two's, 23 fives, and 12 nines?

Solution.

$$\binom{52}{17, 23, 12} ::= \frac{52!}{17!\,23!\,12!}$$

■

Problem 2. **(a)** Show that the Magician could not pull off the trick with a deck larger than 124 cards.

Hint: Compare the number of 5-card hands in an n-card deck with the number of 4-card sequences.

Solution. For a match to be possible with a n-card deck, the number, $\binom{n}{5}$, of 5-card hands must be at most as large as the number, $(n)_4$, of 4-card sequences. So

$$(n)_4(n-4)/5! = \binom{n}{5} \leq (n)_4,$$

which implies

$$n - 4 \leq 5!$$

and hence $n \leq 124$.

■

(b) Show that, in principle, the Magician could pull off the Card Trick with a deck of 124 cards.

Hint: Hall's Theorem and degree-constrained (10.6.5) graphs.

Solution. In principle the trick is possible iff the bipartite graph between 5-card hands and 4-card sequences has a matching for the hands. In this graph, the degree of each hand is $5! = 120$, whatever the size of deck. The degree of each sequence of 4 will be the number of cards remaining in the deck. With a deck of 124, there will be 120 cards remaining, so the degree of each sequence of 4 will also be 120. Hence, the graph is degree-constrained, and so satisfies Hall's condition for a matching. ■

Problem 3.

The Magician can determine the 5th card in a poker hand when his Assisant reveals the other 4 cards. Describe a similar method for determining 2 hidden cards in a hand of 9 cards when your Assisant reveals the other 7 cards.

Solution. Since there must be $\lceil 9/4 \rceil = 3$ cards with the same suit, our collaborator chooses to hide two of them and then use the third one as the first card to be revealed. So this first revealed card fixes the suit of the two hidden cards; it will also be used as the origin for the offset position of the first hidden card. This first hidden card will in turn be used as the origin for the offset of the other hidden card. There are six cards to code the two offset positions. These suffice to code two offsets of size from one to six. That is, our collaborator can choose one of the $3! = 6$ orders in which to reveal the first three cards and thereby tell us the offset position of the first hidden card. Our collaborator can then choose the order of the final three cards to describe the offset position of the second hidden card from the first. Note that the first revealed card must be chosen so that both offsets are ≤ 6; since the sum of the offsets between successive cards ordered in a cycle from Ace to King is 13, it is not possible for more than one offset between successive cards to exceed seven, so this is always possible.

■

Problem 4.
Solve the following counting problems. Define an appropriate mapping (bijective or k-to-1) between a set whose size you know and the set in question.

(a) An independent living group is hosting nine new candidates for membership. Each candidate must be assigned a task: 1 must wash pots, 2 must clean the kitchen, 3 must clean the bathrooms, 1 must clean the common area, and 2 must serve dinner. Write a multinomial coefficient for the number of ways this can be done.

Solution. There is a bijection from sequences containing one P, two K's, three B's, a C, and two D's. In any such sequence, the letter in the ith position specifies the task assigned to the ith candidate. Therefore, the number of possible assignments is:

$$\binom{9}{1,2,3,1,2} ::= \frac{9!}{1!\,2!\,3!\,1!\,2!}$$

■

(b) Write a multinomial coefficient for the number of nonnegative integer solutions for the equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 = 8. \tag{1}$$

Solution. There is a bijection from solutions over $\mathbb{N}$ for (1) to bit strings with eight 0's and four 1's. Namely, letting $\mathtt{0}^x$ represent a string of x zeroes,

$$(x_1, x_2, x_3, x_4, x_5) \in \mathbb{N}^5 \mapsto \mathtt{0}^{x_1}\mathtt{1}\mathtt{0}^{x_2}\mathtt{1}\mathtt{0}^{x_3}\mathtt{1}\mathtt{0}^{x_4}\mathtt{1}\mathtt{0}^{x_5}$$

Therefore, there are

$$\binom{12}{4}$$

nonnegative integer solutions to (1). ■

(c) How many nonnegative integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 17?

Solution. We identify the nonnegative integers less than 1,000,000 with the length 6 strings of decimal digits. Then there is a bijection with pairs:

$$(\text{position of the } 9, \text{successive values of other 5 digits})$$

The sum of the other 5 digits is equal to 8, so the number of ways to choose their values is equal to the number of solutions over the nonnegative integers to (1), namely, $\binom{12}{4}$. So by the product rule there are

$$6 \cdot \binom{12}{4}$$

such integers. ■