


Mathematics for Computer Science
 MIT 6.042J/18.062J


Bookkeeper Rule Counting Magic


 Albert R Meyer, April 14, 2010 lec.10W.1


bookkeeper rule

permutations of the word $\frac{10!}{2!2!3!}$
bookkeeper?

- # perms $bo_1o_2k_1k_2e_1e_2pe_3r = 10!$
- map perm $o_1be_1o_2k_1rk_2e_2pe_3$ to **obeokrkepe**
- 2 o's, 2 k's, 3 e's:
map is $2! \cdot 2! \cdot 3!$ -to-1


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bookkeeper rule

permutations of a word with n_1 a's, n_2 b's, ..., n_k z's is


$$\binom{n}{n_1, n_2, \dots, n_k} ::= \frac{n!}{n_1! n_2! \dots n_k!}$$

multinomial coefficient

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A Magic Trick


audience chooses 5 cards
 Assistant reveals 4 of them
 Magician announces 5th card!

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A Magic Trick

Let's do it!

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Assistant's Choices

Decide the order of the 4 cards:
 $4! = 24$ orderings
 -- but 48 cards remain
 Decide which 4 cards to list

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Map hands to 4-Card lists

5-card hands (no order) → ? → 4-card lists (ordered)

list must come from hand **X**

Which one to pick?

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Map hands to 4-Card lists

5-card hands (no order) → ? → 4-card lists (ordered)

How can we ensure consistency?

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Map hands to 4-Card lists

5-card hands (no order) → 4-card lists (ordered)

Every hand must have an identifying list!

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perfect matching of the hands ...is what we need

5-card hands (no order) → 4-card lists (ordered)

Every hand must have an identifying list!

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Match hands with 4-Card lists

$\text{deg} = 5 \cdot 4 \cdot 3 \cdot 2 = 120$

$\text{deg} = 52 - 4 = 48$

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Match hands with 4-Card lists

So graph is **degree-constrained** and hence has a matching that A & M can use

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A Memorable Matching?

$$\binom{52}{5} = 2,598,960 \text{ hands to match}$$

How will A & M learn any matching this big?

Here's how:



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Magic Trick Revealed (I)

Among 5 cards chosen:
at least 2 have the same suit
(Pigeonhole Principle)

A lists one of them 1st

Aha! The first card has the same suit as the hidden card!



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Magic Trick Revealed (II)

How does M figure out the rank of the hidden card?

Aha! Look at the order of the other 3 cards!



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Magic Trick Revealed (II)

Fix ordering of the deck

$A\clubsuit < A\diamondsuit < A\heartsuit < A\spadesuit <$

$2\clubsuit < 2\diamondsuit < 2\heartsuit < 2\spadesuit <$

\vdots

$K\clubsuit < K\diamondsuit < K\heartsuit < K\spadesuit$



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Magic Trick Revealed (II)

Possible orders for the remaining 3 cards:

{ **SML, SLM, MSL, MLS, LSM, LMS** }



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Magic Trick Revealed (II)

Wait! Only have 6 sequences of the remaining 3 cards, but 12 possible hidden cards of the known suit!

*Of two cards with the same suit, choosing which to reveal can give 1 more bit of information!
Aha!*



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Clockwise Distance

The *smaller clockwise distance* between 2 card ranks is at most **6**:

Hide card with smaller offset.

Reveal the other card

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Magic Trick Revealed (Finally)

- The first card determines the hidden suit (♠ ♥ ♦ ♣).
- Hidden rank (A ... K) = first-card rank + offset (≤ 6).
- Offset given by order of remaining 3 cards:

SML = 1, SLM = 2, MSL = 3, MLS = 4, LSM = 5, LMS = 6.

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Example

First: 10 of Clubs Hidden: King of Spades

Offset = 1 = **SML**: 9 of Hearts, 4 of Diamonds, Queen of Spades

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won't work with 4-card hands

audience can pick	A can reveal
$\binom{52}{4} = 270,725$	$\frac{52!}{49!} = 132,600$
possible 4-card hands	possible 3-card lists

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won't work with 4 card hands

so at least

$$\left\lceil \frac{270,725}{132,600} \right\rceil = 3$$

hands map to the **same list**
- M can't tell which!

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Team Problems

Problems

1 - 4

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