Mathematics for Computer Science

MIT 6.042J/18.062J

Bookkeeper Rule Counting Magic

bookkeeper rule

permutations of the word **bookkeeper** ? $\frac{10!}{2!2!3!}$

- # perms $bo_1o_2k_1k_2e_1e_2pe_3r = 10!$
- map perm $o_1be_1o_2k_1rk_2e_2pe_3$ to **obeokrkepe**
- 2 o's, 2 k's, 3 e's: map is $2!\cdot2!\cdot3!$-to-1

bookkeeper rule

permutations of a word with n_1 **a**'s, n_2 **b**'s, …, n_k **z**'s is

$$\binom{n}{n_1, n_2, \cdots, n_k} ::= \frac{n!}{n_1!n_2!\cdots n_k!}$$

multinomial coefficient

A Magic Trick

audience chooses 5 cards

Assistant reveals 4 of them

Magician announces 5th card!

A Magic Trick

Let's do it!

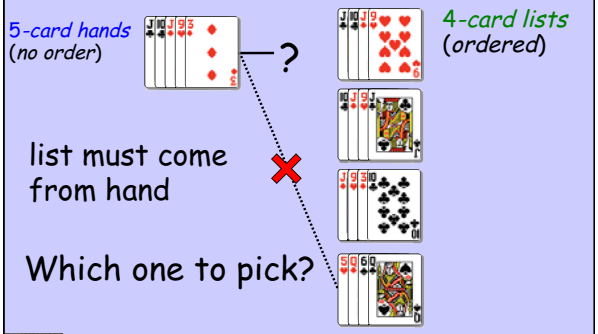
Assistant's Choices

Decide the order of the 4 cards:
4! = 24 orderings

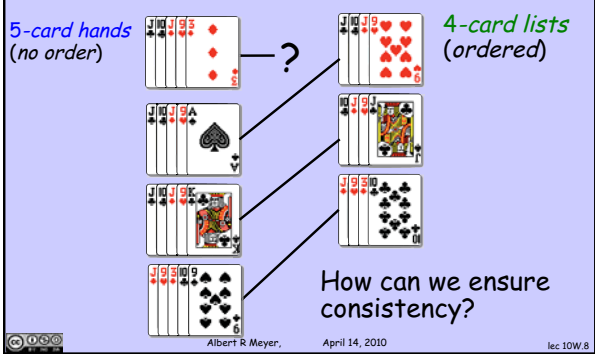
-- but 48 cards remain

Decide which 4 cards to list

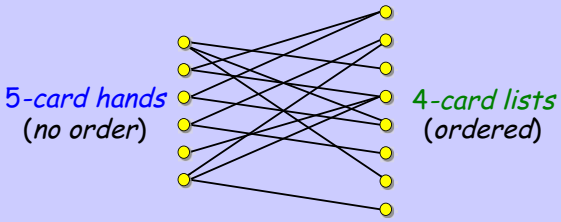
Map hands to 4-Card lists



Map hands to 4-Card lists

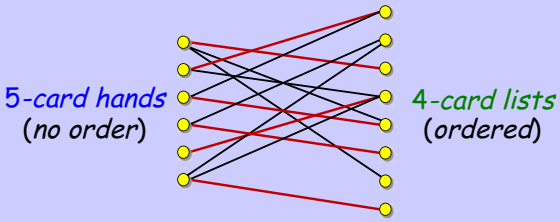


Map hands to 4-Card lists



Every hand must have an identifying list!

perfect matching of the hands …is what we need



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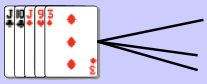
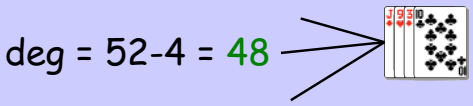
Match hands with 4-Card lists



$\deg = 5 \cdot 4 \cdot 3 \cdot 2 = 120$



$\deg = 52 - 4 = 48$

Match hands with 4-Card lists

So graph is degree-constrained and hence has a matching that A & M can use

A Memorable Matching?

$\binom{52}{5} = 2{,}598{,}960$ hands to match

How will A & M learn any matching this big?

Here's how:

Magic Trick Revealed (I)

Among 5 cards chosen:

at least 2 have the same suit

(*Pigeonhole Principle*)

A lists one of them 1st

Aha! The first card has the same suit as the hidden card!

Magic Trick Revealed (II)

How does M figure out the rank of the hidden card?

Aha! Look at the order of the other 3 cards!

Magic Trick Revealed (II)

Fix ordering of the deck

$A\clubsuit < A\diamondsuit < A\heartsuit < A\spadesuit <$

$2\clubsuit < 2\diamondsuit < 2\heartsuit < 2\spadesuit <$

$\vdots$

$K\clubsuit < K\diamondsuit < K\heartsuit < K\spadesuit$

Magic Trick Revealed (II)

Possible orders for the remaining 3 cards:

{ SML, SLM, MSL, MLS, LSM, LMS }

Magic Trick Revealed (II)

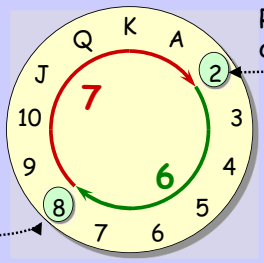
Wait! **Only have 6 sequences of the remaining 3 cards, but 12 possible hidden cards of the known suit!**

*Of **two** cards with the same suit, choosing **which** to reveal can give 1 more bit of information!*

Aha!

Clockwise Distance

The *smaller* clockwise distance between 2 card ranks is at most **6**:



Reveal the other card

Hide card with smaller offset.

Magic Trick Revealed (Finally)

- The first card determines the hidden suit (♠ ♥ ♦ ♣) .
- Hidden rank (A … K) = first-card rank + offset (≤ 6).
- Offset given by order of remaining 3 cards:

SML = 1, **SLM** = 2, **MSL** = 3,
MLS = 4, **LSM** = 5, **LMS** = 6.

Example

same suit

First: 

Hidden: 

Offset = 1 = **SML**: 

won't work with 4-card hands

audience can pick

$\binom{52}{4} = 270{,}725$

possible 4-card hands

A can reveal

$\frac{52!}{49!} = 132{,}600$

possible 3-card lists

won't work with 4 card hands

so at least

$$\left\lceil \frac{270{,}725}{132{,}600} \right\rceil = 3$$

hands map to the same list
– M can't tell which!

Team Problems

Problems

1 — 4

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Spring 2010