Solutions to In-Class Problems Week 10, Mon.

Problem 1.
Solve the following problems using the pigeonhole principle. For each problem, try to identify the *pigeons*, the *pigeonholes*, and a *rule* assigning each pigeon to a pigeonhole.

(a) Every MIT ID number starts with a 9 (we think). Suppose that each of the 75 students in 6.042 sums the nine digits of his or her ID number. Explain why two people must arrive at the same sum.

Solution. The students are the pigeons, the possible sums are the pigeonholes, and we map each student to the sum of the digits in his or her MIT ID number. Every sum is in the range from $9 + 8 \cdot 0 = 9$ to $9 + 8 \cdot 9 = 81$, which means that there are 73 pigeonholes. Since there are more pigeons than pigeonholes, there must be two pigeons in the same pigeonhole; in other words, there must be two students with the same sum. ∎

(b) In every set of 100 integers, there exist two whose difference is a multiple of 37.

Solution. The pigeons are the 100 integers. The pigeonholes are the numbers 0 to 36. Map integer k to $\mathrm{rem}(k, 37)$. Since there are 100 pigeons and only 37 pigeonholes, two pigeons must go in the same pigeonhole. This means $\mathrm{rem}(k_1, 37) = \mathrm{rem}(k_2, 37,)$, which implies that $k_1 - k_2$ is a multiple of 37. ∎

(c) For any five points inside a unit square (not on the boundary), there are two points at distance *less than* $1/\sqrt{2}$.

Solution. The pigeons are the points. The pigeonholes are the four subsquares of the unit square, each of side length $1/2$.

Pigeons are assigned to the subsquare that contains them, except that if the pigeon is on a boundary, it gets assigned to the leftmost and then lowest possible subsquare that includes it (so the point at $(1/2, 1/2)$ is assigned to the lower left subsquare).

There are five pigeons and four pigeonholes, so more than one point must be in the same subsquare. The diagonal of a subsquare is $1/\sqrt{2}$, so two pigeons in the same hole are at most this distance. But pigeons must be inside the unit square, so two pigeons cannot be at the opposite ends of the same subsquare diagonal. So at least one of them must be inside the subsquare, so their distance is less than the length of the diagonal. ∎

(d) Show that if $n + 1$ numbers are selected from $\{1, 2, 3, \ldots, 2n\}$, two must be consecutive, that is, equal to k and $k + 1$ for some k.

Solution. The pigeonholes will be the n sets $\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{2n - 1, 2n\}$. The pigeons will be the $n + 1$ selected numbers. A pigeon is assigned to the unique pigeon hole of which it is a member. By the Pigeonhole Principle, two pigeons must assigned to some hole, and these are the two consecutive numbers required. Notice that we've actually shown a bit more: there will be two consecutive numbers with the smaller being odd.   ∎

Problem 2.
Answer the following quesions using the Generalized Product Rule.

(a) Next week, I'm going to get really fit! On day 1, I'll exercise for 5 minutes. On each subsequent day, I'll exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be 5, 6, 9, 9, 9, 11, 12. How many such sequences are possible?

Solution. The number of minutes on the first day can be selected in 1 way. The number of minutes on each subsequent day can be selected in 4 ways. Therefore, the number of exercise sequences is $1 \cdot 4^6$ by the extended product rule.   ∎

(b) An *r-permutation* of a set is a sequence of r distinct elements of that set. For example, here are all the 2-permutations of $\{a, b, c, d\}$:

$$
\begin{array}{lll}
(a, b) & (a, c) & (a, d) \\
(b, a) & (b, c) & (b, d) \\
(c, a) & (c, b) & (c, d) \\
(d, a) & (d, b) & (d, c)
\end{array}
$$

How many r-permutations of an n-element set are there? Express your answer using factorial notation.

Solution. There are n ways to choose the first element, $n - 1$ ways to choose the second, $n - 2$ ways to choose the third, $\ldots$, and $n - r + 1$ ways to choose the r-th element. Thus, there are:

$$
n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1) = \frac{n!}{(n - r)!}
$$

r-permutations of an n-element set.   ∎

(c) How many $n \times n$ matrices are there with *distinct* entries drawn from $\{1, \ldots, p\}$, where $p \geq n^2$?

Solution. There are p ways to choose the first entry, $p - 1$ ways to choose the second for each way of choosing the first, $p - 2$ ways of choosing the third, and so forth. In all there are

$$
p(p - 1)(p - 2) \cdots (p - n^2 + 1) = \frac{p!}{(p - n^2)!}
$$

such matrices. Alternatively, this is the number of n^2-permutations of a p element set, which is $p!/(p - n^2)!$.   ∎

Problem 3.
Your 6.006 tutorial has 12 students, who are supposed to break up into 4 groups of 3 students each. Your TA has observed that the students waste too much time trying to form balanced groups, so he decided to pre-assign students to groups and email the group assignments to his students.

(a) Your TA has a list of the 12 students in front of him, so he divides the list into consecutive groups of 3. For example, if the list is ABCDEFGHIJKL, the TA would define a sequence of four groups to be $(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\})$. This way of forming groups defines a mapping from a list of twelve students to a sequence of four groups. This is a k-to-1 mapping for what k?

Solution. Two lists map to the same sequence of groups iff the first 3 students are the same on both lists, and likewise for the second, third, and fourth consecutive sublists of 3 students. So for a given sequence of 4 groups, the number of lists which map to it is

$$(3!)^4$$

because there are 3! ways to order the students in each of the 4 consecutive sublists. ∎

(b) A group assignment specifies which students are in the same group, but not any order in which the groups should be listed. If we map a sequence of 4 groups,

$$(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}),$$

into a group assignment

$$\{\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}\},$$

this mapping is j-to-1 for what j?

Solution. $4!$.

Each of the 4! sequences of a particular set of four groups maps to that set of groups. ∎

(c) How many group assignments are possible?

Solution.
$$\frac{12!}{4! \cdot (3!)^4} = 15400$$

different assignments.

There are 12! possible lists of students, and we can map each list to an assignment by first mapping the list to a sequence of four groups, and then mapping the sequence to the assignment. Since the first map is $(3!)^4$-to-1 and and the second is 4!-to-1, the composite map is $(3!)^4 \cdot 4!$-to-1. So by the Division Rule, $12! = ((3!)^4 \cdot 4!) A$ where A is the number of assignments. ∎

(d) In how many ways can $3n$ students be broken up into n groups of 3?

Solution.

$$\frac{(3n)!}{(3!)^n n!}.$$

This follows simply by replacing "12" by "$3n$" and "4" by "n"in the solution to the previous problem parts. ∎

Problem 4.
A pizza house is having a promotional sale. Their commercial reads:

> We offer 9 different toppings for your pizza! Buy 3 large pizzas at the regular price, and you can get each one with as many different toppings as you wish, absolutely free. That's $22,369,621$ different ways to choose your pizzas!

The ad writer was a former Harvard student who had evaluated the formula $(2^9)^3/3!$ on his calculator and gotten close to $22,369,621$. Unfortunately, $(2^9)^3/3!$ is obviously not an integer, so clearly something is wrong. What mistaken reasoning might have led the ad writer to this formula? Explain how to fix the mistake and get a correct formula.

Solution. The number of ways to choose toppings for one pizza is the number of the possible subsets of the set of 9 toppings, namely, 2^9. The ad writer presumably then used the Product Rule to conclude that there were $(2^9)^3$ *sequences* of three topping choices. Then he probably reasoned that each way of making three topping choices arises from 3! sequences, so the Division Rule would imply that the number of ways to choose three pizzas is $(2^9)^3/3!$.

It's true that every set of three *different* topping choices arises from 3! different length-3 sequences of choices. The mistake is that if some of the three choices are the same, then the set of three choices arises from *fewer* than 3! sequences. For example, if all three pizzas have the same toppings, there is only one sequence of topping choices for them.

One fix is to consider ways to choose toppings with 1, 2 and 3 different topping choices. There are $2^9(2^9 - 1)(2^9 - 2)/3!$ ways to choose a set of 3 different choices, $2^9(2^9 - 1)$ ways to choose one topping choice to be used on two pizzas and a second choice for the third pizza, and 2^9 ways to choose one topping for all three pizzas, giving

$$\frac{2^9(2^9 - 1)(2^9 - 2)}{3!} + 2^9(2^9 - 1) + 2^9 = 22,500,864.$$

ways to choose three pizzas.

Alternatively, we can observe that this is exactly the problem of selecting a dozen donuts of five possible different kinds – except now there are 3 donuts and 2^9 kinds. Hence, there is a bijection to the number of $(2^9 + 2)$-bit strings with exactly $2^9 - 1$ ones and 3 zeros:

$$\binom{2^9 + 2}{3} = 22,500,864.$$

∎

6.042J / 18.062J Mathematics for Computer Science
Spring 2010