

**Mathematics for Computer Science**  
 MIT 6.042J/18.062J

# Generalized Counting Rules


Albert R Meyer, April 12, 2010 lec 10M.1


**Pigeonhole Principle**


Mapping Rule: **total injection** from  $A$  to  $B$  implies  $|A| \leq |B|$ .

If  $|A| > |B|$ , then **no total injection** from  $A$  to  $B$ .


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**Pigeonhole Principle**


If **more pigeons** than pigeonholes,




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**Pigeonhole Principle**

then **some hole** must have  $\geq$  **two pigeons!**



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**example: 5 Card Draw**

set of 5 cards: must have  $\geq 2$  with the **same suit**.



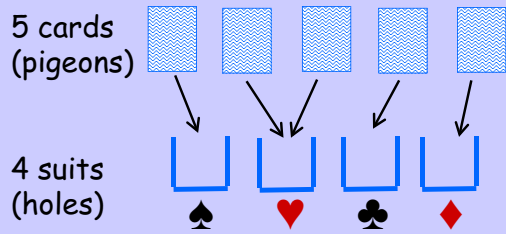


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**5 Card Draw**

5 cards (pigeons)



4 suits (holes)

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## 10 Card Draw

10 cards: how many have the same suit?

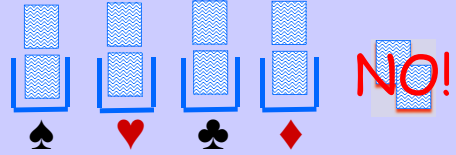


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## 10 Card Draw



< 3 cards in every hole?



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## 10 Card Draw

# cards with same suit

$$\geq \left\lceil \frac{10}{4} \right\rceil = 3$$

"ceiling," means round up



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## Generalized Pigeonhole Principle

If  $n$  pigeons and  $h$  holes, then some hole has  $\geq$

$$\left\lceil \frac{n}{h} \right\rceil \text{ pigeons.}$$



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## Generalized Product Rule

# lineups of 5 students in 6.042? let  $S ::= 6.042$  students

$|S| = 91$  so

~~|lineups of 5 students|~~ NO!

lineups have no repeats:

|seqs in  $S^5$  with no repeats| ?



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## Generalized Product Rule

|seqs in  $S^5$  with no repeats|

91 choices for 1<sup>st</sup> student,

90 choices for 2<sup>nd</sup> student,

89 choices for 3<sup>rd</sup> student,

88 choices for 4<sup>th</sup> student,

87 choices for 5<sup>th</sup> student

$$= 91 \cdot 90 \cdot 89 \cdot 88 \cdot 87 = \frac{91!}{86!}$$



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### Generalized Product Rule

Q a set of length- $k$  sequences  
 if  $n_1$  possible 1<sup>st</sup> elements,  
 $n_2$  possible 2<sup>nd</sup> elements  
 (for each first entry),  
 $n_3$  possible 3<sup>rd</sup> elements  
 (for each 1<sup>st</sup> & 2<sup>nd</sup> entry,...)

then,  $|Q| = n_1 \cdot n_2 \cdot \dots \cdot n_k$



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### Division Rule

$$\frac{\#6.042 \text{ students} = \#6.042 \text{ students' fingers}}{10}$$



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### Division Rule

if function from  $A$  to  $B$   
 is  $k$ -to-1, then

$$|A| = k|B|$$

(generalized Bijection Rule)



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### Counting Subsets

How many size 4 subsets of  $\{1,2,\dots,13\}$ ?

Let  $A ::=$  permutations of  $\{1,2,\dots,13\}$

$B ::=$  size 4 subsets

map  $a_1 a_2 a_3 a_4 a_5 \dots a_{12} a_{13} \in A$   
 to  $\{a_1, a_2, a_3, a_4\} \in B$



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### Counting Subsets

$a_1 a_3 a_2 a_4 a_5 \dots a_{12} a_{13}$  also maps  
 to  $\{a_1, a_2, a_3, a_4\}$

so does  $a_1 a_3 a_2 a_4 a_{13} \dots a_{12} a_5$   
 $4!$  perms  $9!$  perms

all map to same set

$$4! \cdot 9! \text{-to-1}$$



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### Counting Subsets

$$13! = |A| = (4! \cdot 9!) |B|$$


so # of size 4 subsets is

$$\binom{13}{4} ::= \frac{13!}{4!9!}$$




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 **Counting Subsets**

#  $m$  element subsets  
of an  $n$  element set is

$$\binom{n}{m} ::= \frac{n!}{m!(n-m)!}$$

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 **counting 2-pair poker hands**

a 2-pair hand has

- 2 cards of some rank
- 2 cards of a second rank
- 1 card of still a third rank

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 **counting 2-pair poker hands**

a 2-pair hand:

$K\spadesuit, K\heartsuit, A\clubsuit, A\spadesuit, 3\heartsuit$

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 **counting 2-pair poker hands**

to count, choose:

- 1<sup>st</sup> pair rank (13 ranks)
- 2<sup>nd</sup> pair rank (12 ranks left)
- last card rank (11 ranks left)

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 **counting 2-pair poker hands**

then choose:

- 1<sup>st</sup> pair suits  $\binom{4}{2}$  sets of 2 suits
- 2<sup>nd</sup> pair suits  $\binom{4}{2}$  sets of 2 suits
- last card suit (4 suits)

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 **counting 2-pair poker hands**

successively choosing:

$K, A, 3, \{\heartsuit, \spadesuit\}, \{\clubsuit, \diamondsuit\}, \clubsuit$

specifies 2-pair hand:

$K\spadesuit, K\heartsuit, A\clubsuit, A\spadesuit, 3\heartsuit$

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counting 2-pair poker hands  
so # 2-pair hands is

$$13 \cdot 12 \cdot 11 \cdot \binom{4}{2}! \cdot \binom{4}{2} \cdot 4$$

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counting 2-pair poker hands  
this method counts 6-tuples  
[1<sup>st</sup> card ranks] × [2<sup>nd</sup> card ranks]  
× [last card rank] ×  
[1<sup>st</sup> card suits] × [2<sup>nd</sup> card suits]  
× [last card suit]  
**correctly**

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counting 2-pair poker hands  
but the correspondence to  
2-pair hands is **not a bijection**:

(K, A, 3, {♥, ♦}, {♦, ♠}, ♣)

→ K♦, K♥, A♦, A♠, 3♣

(A, K, 3, {♦, ♠}, {♥, ♦}, ♣)

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counting 2-pair poker hands  
to count, choose: **the bug**

- 1<sup>st</sup> pair rank (13 ranks)
- 2<sup>nd</sup> pair rank (12 ranks left)
- last card rank (11 ranks left)

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counting 2-pair poker hands  
to count, choose: **the bug**

- 1<sup>st</sup> pair rank (13 ranks)
- 2<sup>nd</sup> pair rank (12 ranks left)
- last card rank (11 ranks left)

**either pair might be 1<sup>st</sup>**

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counting 2-pair poker hands  
map from 6-tuples  
(K, A, 3, {♥, ♦}, {♦, ♠}, ♣)  
to 2-pair hands  
K♦, K♥, A♦, A♠, 3♣  
**is 2-to-1**

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counting 2-pair poker hands  
so # 2-pair hands is

$$13 \cdot 12 \cdot 11 \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 4$$

NO!



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counting 2-pair poker hands  
so # 2-pair hands is **really**

$$\frac{1}{2} \cdot 13 \cdot 12 \cdot 11 \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 4$$



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Team Problems

Problems  
1–4



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