Mathematics for Computer Science

MIT 6.042J/18.062J

Generalized Counting Rules

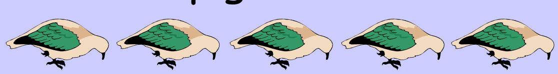
Pigeonhole Principle

Mapping Rule: total injection from A to B implies $|A| \le |B|$.

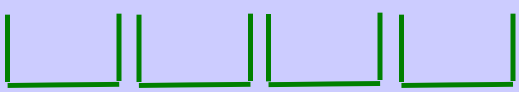
If $|A| > |B|$, then no total injection from A to B.

Pigeonhole Principle

If more pigeons



than pigeonholes,



Pigeonhole Principle

then some hole must have $\ge$ two pigeons!

example: 5 Card Draw

set of 5 cards: must have ≥ 2 with the same suit.

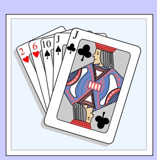
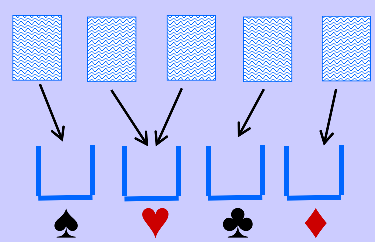


Image by MIT OpenCourseWare.

5 Card Draw

5 cards (pigeons)

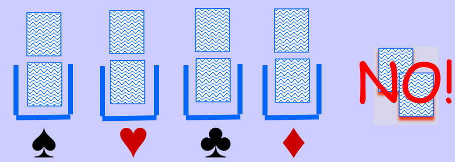
4 suits (holes)



10 Card Draw

10 cards: how many have the same suit?

10 Card Draw



< 3 cards in every hole?

10 Card Draw

cards with same suit

$$\geq \left\lceil \frac{10}{4} \right\rceil = 3$$

"ceiling," means round up

Generalized Pigeonhole Principle

If n pigeons and h holes, then some hole has $\geq$

$$\left\lceil \frac{n}{h} \right\rceil$$ pigeons.

Generalized Product Rule

lineups of 5 students in 6.042? let S::= 6.042 students

$|S| = 91$ so

~~|lineups of 5 students|~~ NO!

lineups have no repeats:

|seqs in S^5 with no repeats| ?

Generalized Product Rule

|seqs in S^5 with no repeats|

91 choices for 1st student,

90 choices for 2nd student,

89 choices for 3rd student,

88 choices for 4th student,

87 choices for 5th student

$$= 91 \cdot 90 \cdot 89 \cdot 88 \cdot 87 = \frac{91!}{86!}$$

Generalized Product Rule

Q a set of length-k sequences

if n_1 possible 1st elements,
n_2 possible 2nd elements (for each first entry),
n_3 possible 3rd elements (for each 1st & 2nd entry,...)

then, $|Q| = n_1 \cdot n_2 \cdots n_k$

Division Rule

$$\#6.042 \text{ students} = \frac{\#6.042 \text{ students' fingers}}{10}$$

Division Rule

if function from A to B is k-to-1, then

$$|A| = k|B|$$

(generalized Bijection Rule)

Counting Subsets

How many size 4 subsets of {1,2,…,13}?

Let A::= permutations of {1,2,…,13}
B::= size 4 subsets

map $a_1 a_2 a_3 a_4 a_5 \ldots a_{12} a_{13} \in A$

to $\{a_1, a_2, a_3, a_4\} \in B$

Counting Subsets

$a_1 a_3 a_2 a_4 a_5 \ldots a_{12} a_{13}$ also maps to $\{a_1, a_2, a_3, a_4\}$

so does $\underbrace{a_1 a_3 a_2 a_4}_{4! \text{ perms}} \underbrace{a_{13} \ldots a_{12} a_5}_{9! \text{ perms}}$

all map to same set

$4! \cdot 9!$-to-1

Counting Subsets

$$13! = |A| = (4! \cdot 9!)|B|$$

so # of size 4 subsets is

$$\binom{13}{4} ::= \frac{13!}{4!9!}$$

Counting Subsets

m element subsets of an n element set is

$$\binom{n}{m} ::= \frac{n!}{m!(n-m)!}$$

counting 2-pair poker hands

a 2-pair hand has

- 2 cards of some rank
- 2 cards of a second rank
- 1 card of still a third rank

counting 2-pair poker hands

a 2-pair hand:

K♦, K♥, A♦, A♠, 3♣

counting 2-pair poker hands

to count, choose:

- 1^{st} pair rank (13 ranks)
- 2^{nd} pair rank (12 ranks left)
- last card rank (11 ranks left)

counting 2-pair poker hands

then choose:

- 1^{st} pair suits $\binom{4}{2}$ sets of 2 suits
- 2^{nd} pair suits $\binom{4}{2}$ sets of 2 suits
- last card suit (4 suits)

counting 2-pair poker hands

successively choosing:

K, A, 3, {♥, ♦}, {♦, ♠}, ♣

specifies 2-pair hand:

K♦, K♥, A♦, A♠, 3♣

counting 2-pair poker hands

so # 2-pair hands is

$$13 \cdot 12 \cdot 11 \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 4$$

NO!

counting 2-pair poker hands

this method counts 6-tuples

[1st card ranks] × [2nd card ranks] × [last card rank] ×

[1st card suits] × [2nd card suits] × [last card suit]

correctly

counting 2-pair poker hands

but the correspondence to 2-pair hands is not a bijection:

(K, A, 3, {♥, ♦}, {♦, ♠}, ♣)

K♦, K♥, A♦, A♠, 3♣

(A, K, 3, {♦, ♠}, {♥, ♦}, ♣)

counting 2-pair poker hands

to count, choose: the bug

- 1st pair rank (13 ranks)
- 2nd pair rank (12 ranks left)
- last card rank (11 ranks left)

counting 2-pair poker hands

to count, choose: the bug

- 1st pair rank (13 ranks)
- 2nd pair rank (12 ranks left)
- last card rank (11 ranks left)

either pair might be 1st

counting 2-pair poker hands

map from 6-tuples

(K, A, 3, {♥, ♦}, {♦, ♠}, ♣)

to 2-pair hands

K♦, K♥, A♦, A♠, 3♣

is 2-to-1

counting 2-pair poker hands

so # 2-pair hands is

$$13 \cdot 12 \cdot 11 \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 4$$

NO!

counting 2-pair poker hands

so # 2-pair hands is really

$$\frac{1}{2} \cdot 13 \cdot 12 \cdot 11 \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 4$$

Team Problems

Problems 1—4