

Solutions to In-Class Problems Week 9, Fri.

Problem 1.

A license plate consists of either:

- 3 letters followed by 3 digits (standard plate)
- 5 letters (vanity plate)
- 2 characters – letters or numbers (big shot plate)

Let L be the set of all possible license plates.

(a) Express L in terms of

$$\begin{aligned}\mathcal{A} &= \{A, B, C, \dots, Z\} \\ \mathcal{D} &= \{0, 1, 2, \dots, 9\}\end{aligned}$$

using unions (\cup) and set products (\times).

Solution.

$$L = (A^3 \times D^3) \cup A^5 \cup (A \cup D)^2$$

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(b) Compute $|L|$, the number of different license plates, using the sum and product rules.

Solution.

$$\begin{aligned}|L| &= |(A^3 \times D^3) \cup A^5 \cup (A \cup D)^2| \\ &= |(A^3 \times D^3)| + |A^5| + |(A \cup D)^2| && \text{Sum Rule} \\ &= |A|^3 \cdot |D|^3 + |A|^5 + |A \cup D|^2 && \text{Product Rule} \\ &= |A|^3 \cdot |D|^3 + |A|^5 + (|A| + |D|)^2 && \text{Sum Rule} \\ &= 26^3 \cdot 10^3 + 26^5 + 36^2 = 29458672\end{aligned}$$

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Problem 2.

An n -vertex *numbered tree* is a tree whose vertex set is $\{1, 2, \dots, n\}$ for some $n > 2$. We define the *code* of the numbered tree to be a sequence of $n - 2$ integers from 1 to n obtained by the following recursive process:

If there are more than two vertices left, write down the *father* of the largest leaf^a, delete this *leaf*, and continue this process on the resulting smaller tree.

If there are only two vertices left, then stop —the code is complete.

^aThe necessarily unique node adjacent to a leaf is called its *father*.

For example, the codes of a couple of numbered trees are shown in the Figure 1.

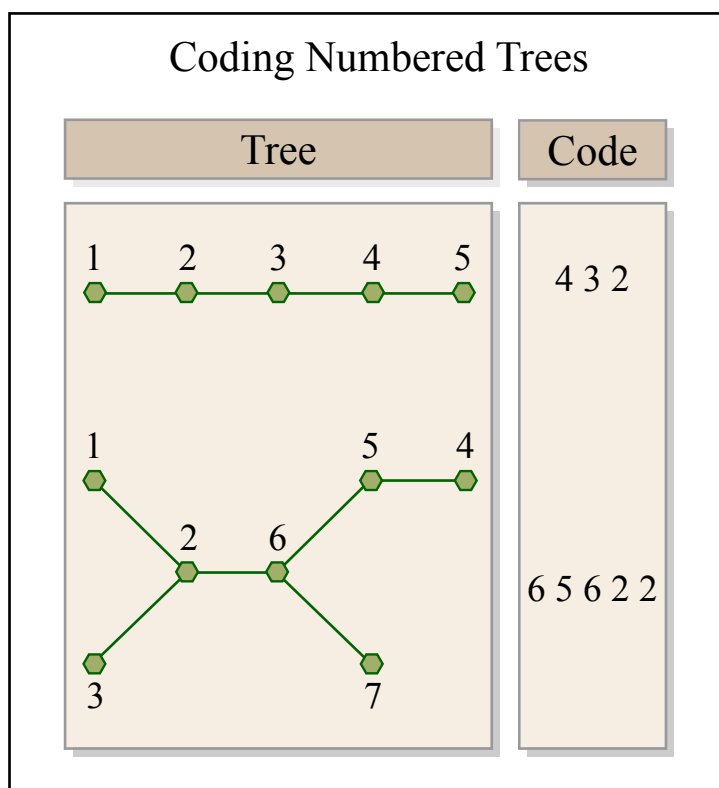


Image by MIT OpenCourseWare.

Figure 1:

(a) Describe a procedure for reconstructing a numbered tree from its code.

Solution. The key observation is that, given a code of length $n - 2$, the numbers between 1 and n which *do not appear* in the code are precisely the leaves of the tree. This follows because the vertices left at the end of the process are both leaves. So the procedure must have changed all the nonleaf vertices into leaves, and this implies that all the nonleaf vertices appear in the code.

Hence, the largest missing number is a leaf attached to the first number of the code. The rest of the tree can now be reconstructed by deleting the first number in the code, henceforth ignoring

the largest leaf, and proceeding recursively on the rest of the code. (We're using the obvious fact that what's left after deleting a leaf from a tree is another tree.)

More precisely, the reconstruction procedure applies to any finite tree whose vertex set is totally ordered. The procedure takes *two* parameters: the vertex set, V , and a length $|V| - 2$ "code" sequence, S , of elements in V . If l is the largest element in V which does not appear in S , and f is the first element of S , then the reconstructed tree is obtained by adding edge (l, f) to the tree reconstructed by calling the procedure recursively with first argument $V - \{l\}$ and second argument equal to the code obtained by erasing the initial f from S . The procedure terminates when $|V| = 2$, returning the edge between the two numbers in V .

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(b) Conclude there is a bijection between the n -vertex numbered trees and $\{1, \dots, n\}^{n-2}$, and state how many n -vertex numbered trees there are.

Solution. There are exactly as many n -vertex numbered trees as the number of possible code words, that is, the number of length $n - 2$ sequences of integers between 1 and n . So there are n^{n-2} numbered trees.

The reason is that the map from trees to codes is a bijection. To see this, note that the tree reconstruction procedure finds *the only possible tree* with that code. So there can't be two trees with the same code, that is, the map from a tree to its code is an injection. But since the reconstruction procedure finds a tree for every possible codeword, the map from trees to codes is also a surjection.

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Problem 3. (a) How many of the billion numbers in the range from 1 to 10^9 contain the digit 1? (*Hint:* How many don't?)

Solution. We can count up how many *do not* contain the digit 1 and subtract. So (total number) - (number without 1's) = $10^9 - (9^9 - 1) = 612, 579, 512$ (the -1 is for 0 which is not in our range). ■

(b) There are 20 books arranged in a row on a shelf. Describe a bijection between ways of choosing 6 of these books so that no two adjacent books are selected and 15-bit strings with exactly 6 ones.

Solution. A selection of six among twenty books on a shelf corresponds in an obvious way to a 20-bit string with exactly six 1's. For example, the 20-bit string with 1's in exactly the 3rd, 4th, 5th, 10th, 19th and 20th positions corresponds to selecting 3rd, 4th, 5th, 10th, 19th and 20th books on the shelf.

So the problem reduces to finding a bijection between 20-bit strings with six *nonadjacent* 1's and 15-bit strings with six 1's.

But in a string, s , with six nonadjacent 1's, all but the last 1 must have a 0 to its right. So we can map s to a string with six 1's and five fewer 0's by erasing the 0's immediately to the right of each of the first five 1's. For example, erasing the underlined 0's in the 20-bit string 00010100101000001010 yields the 15-bit string 000110110000110.

This map is a bijection because given any 15-bit string with six 1's, there is a unique 20-bit string with nonadjacent 1's that maps to it, namely, the string obtained by replacing each of the first five 1's in the 15-bit string by a 10. ■

Problem 4.

(a) Let $\mathcal{S}_{n,k}$ be the possible nonnegative integer solutions to the inequality

$$x_1 + x_2 + \cdots + x_k \leq n. \quad (1)$$

That is

$$\mathcal{S}_{n,k} ::= \left\{ (x_1, x_2, \dots, x_k) \in \mathbb{N}^k \mid (1) \text{ is true} \right\}.$$

Describe a bijection between $\mathcal{S}_{n,k}$ and the set of binary strings with n zeroes and k ones.

Solution. The notation 0^x indicates a length x string of 0's.

$$(x_1, x_2, \dots, x_k) \longleftrightarrow 0^{x_1} 1 0^{x_2} 1 \dots 0^{x_k} 1 0^{n-s},$$

where $s ::= \sum_{i=1}^k x_i$. ■

(b) Let $\mathcal{L}_{n,k}$ be the length k weakly increasing sequences of nonnegative integers $\leq n$. That is

$$\mathcal{L}_{n,k} ::= \left\{ (y_1, y_2, \dots, y_k) \in \mathbb{N}^k \mid y_1 \leq y_2 \leq \cdots \leq y_k \leq n \right\}.$$

Describe a bijection between $\mathcal{L}_{n,k}$ and $\mathcal{S}_{n,k}$.

Solution. $(y_1, y_2, \dots, y_k) \longleftrightarrow (y_1, y_2 - y_1, y_3 - y_2, \dots, y_k - y_{k-1})$.

In the other direction,

$$(x_1, x_2, \dots, x_k) \longleftrightarrow (x_1, x_1 + x_2, x_1 + x_2 + x_3, \dots, \sum_{i=1}^k x_i).$$

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