Solutions to In-Class Problems Week 9, Wed.

Problem 1.
Recall that for functions f, g on $\mathbb{N}$, $f = O(g)$ iff

$$\exists c \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \forall n \geq n_0 \quad c \cdot g(n) \geq |f(n)|\,. \tag{1}$$

For each pair of functions below, determine whether $f = O(g)$ and whether $g = O(f)$. In cases where one function is O() of the other, indicate the *smallest nonnegative integer*, c, and for that smallest c, the *smallest corresponding nonnegative integer* n_0 ensuring that condition (1) applies.

(a) $f(n) = n^2, g(n) = 3n$.

$f = O(g)$ YES NO If YES, $c =$ _______, n_0 = _______

Solution. NO. ■

$g = O(f)$ YES NO If YES, $c =$ _______, n_0 = _______

Solution. YES, with $c = 1$, $n_0 = 3$, which works because $3^2 = 9$, $3 \cdot 3 = 9$. ■

(b) $f(n) = (3n - 7)/(n + 4), g(n) = 4$

$f = O(g)$ YES NO If YES, $c =$ _______, n_0 = _______

Solution. YES, with $c = 1, n_0 = 0$ (because $|f(n)| < 3$). ■

$g = O(f)$ YES NO If YES, $c =$ _______, n_0 = _______

Solution. YES, with $c = 2, n_0 = 15$.

Since $\lim_{n\to\infty} f(n) = 3$, the smallest possible c is 2. For $c = 2$, the smallest possible $n_0 = 15$ which follows from the requirement that $2f(n_0) \geq 4$. ■

(c) $f(n) = 1 + (n \sin(n\pi/2))^2, g(n) = 3n$

$f = O(g)$ YES NO If yes, $c =$ _______ n_0 = _______

Solution. NO, because $f(2n) = 1$, which rules out $g = O(f)$ since $g = \Theta(n)$. ■

$g = O(f)$ YES NO If yes, $c =$ _______ n_0 = _______

Solution. NO, because $f(2n + 1) = n^2 + 1 \neq O(n)$ which rules out $f = O(g)$. ■

Problem 2.

(a) Define a function $f(n)$ such that $f = \Theta(n^2)$ and $\text{NOT}(f \sim n^2)$.

Solution. Let $f(n) ::= 2n^2$. ■

(b) Define a function $g(n)$ such that $g = O(n^2)$, $g \neq \Theta(n^2)$ and $g \neq o(n^2)$.

Solution. Let $g(n) ::= (n \sin(n\pi/2))^2 + n\,(\cos(n\pi/2))^2$.

For odd n, we have $g(n) = n^2$, which implies that $g \neq o(n^2)$. For even n, we have $g(n) = n$, which implies $n^2 \neq O(g)$ and hence $g \neq \Theta(n^2)$.

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Problem 3.

False Claim.

$$2^n = O(1). \tag{2}$$

Explain why the claim is false. Then identify and explain the mistake in the following bogus proof.

Bogus proof. The proof by induction on n where the induction hypothesis, $P(n)$, is the assertion (2).

base case: $P(0)$ holds trivially.

inductive step: We may assume $P(n)$, so there is a constant $c > 0$ such that $2^n \leq c \cdot 1$. Therefore,

$$2^{n+1} = 2 \cdot 2^n \leq (2c) \cdot 1,$$

which implies that $2^{n+1} = O(1)$. That is, $P(n+1)$ holds, which completes the proof of the inductive step.

We conclude by induction that $2^n = O(1)$ for all n. That is, the exponential function is bounded by a constant.

■

Solution. A function is $O(1)$ iff it is bounded by a constant, and since the function 2^n grows unboundedly with n, it is not $O(1)$.

The mistake in the bogus proof is in its misinterpretation of the expression 2^n in assertion (2). The intended interpration of (2) is

$$\text{Let } f \text{ be the function defined by the rule } f(n) ::= 2^n. \text{ Then } f = O(1). \tag{3}$$

But the bogus proof treats (2) as an assertion, $P(n)$, about n. Namely, it misinterprets (2) as meaning:

Let f_n be the constant function equal to 2^n. That is, $f_n(k) ::= 2^n$ for all $k \in \mathbb{N}$. Then

$$f_n = O(1). \tag{4}$$

Now (4) is true since every constant function is $O(1)$, and the bogus proof is an unnecessarily complicated, but *correct*, proof that that for each n, the constant function f_n is $O(1)$. But in the last line, the bogus proof switches from the misinterpretation (4) and claims to have proved (3).

So you could say that the exact place where the proof goes wrong is in its first line, where it defines $P(n)$ based on misinterpretation (4). Alternatively, you could say that the proof was a correct proof (of the misinterpretation), and its first mistake was in its last line, when it switches from the misinterpretation to the proper interpretation (3). ■

Problem 4.
Give an elementary proof (without appealing to Stirling's formula) that $\log(n!) = \Theta(n \log n)$.

Solution. One elementary proof goes as follows:

First,

$$\log(n!) = \sum_{i=1}^{n} \log i < \sum_{i=1}^{n} \log n = n \log n.$$

On the other hand,

$$\begin{aligned}
\log(n!) = \sum_{i=1}^{n} \log i &> \sum_{i=\lceil (n+1)/2 \rceil}^{n} \log i \\
&> \sum_{i=\lceil (n+1)/2 \rceil}^{n} \log(n/2) > \frac{n}{2} \cdot \log(n/2) \\
&= \frac{n((\log n) - 1)}{2} = \frac{n \log n}{2} - \frac{n}{2} \\
&> \frac{n \log n}{2} - \frac{n \log n}{6} && \text{for } n > 8. \\
&= \frac{1}{3} \cdot n \log n.
\end{aligned}$$

Therefore, $(1/3) n \log n < \log(n!) < n \log n$ for $n > 8$, proving that $\log(n!) = \Theta(n \log n)$.

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