In-Class Problems Week 9, Wed.

Problem 1.
Recall that for functions f, g on $\mathbb{N}$, $f = O(g)$ iff

$$\exists c \in \mathbb{N}\, \exists n_0 \in \mathbb{N}\, \forall n \geq n_0 \quad c \cdot g(n) \geq |f(n)|\,. \tag{1}$$

For each pair of functions below, determine whether $f = O(g)$ and whether $g = O(f)$. In cases where one function is O() of the other, indicate the *smallest nonnegative integer*, c, and for that smallest c, the *smallest corresponding nonnegative integer* n_0 ensuring that condition (1) applies.

(a) $f(n) = n^2, g(n) = 3n$.

$f = O(g)$ YES NO If YES, $c =$ ______, n_0 = ______

$g = O(f)$ YES NO If YES, $c =$ ______, n_0 = ______

(b) $f(n) = (3n - 7)/(n + 4), g(n) = 4$

$f = O(g)$ YES NO If YES, $c =$ ______, n_0 = ______

$g = O(f)$ YES NO If YES, $c =$ ______, n_0 = ______

(c) $f(n) = 1 + (n \sin(n\pi/2))^2, g(n) = 3n$

$f = O(g)$ YES NO If yes, $c =$ ______ n_0 = ______

$g = O(f)$ YES NO If yes, $c =$ ______ n_0 = ______

Problem 2.

(a) Define a function $f(n)$ such that $f = \Theta(n^2)$ and $\text{NOT}(f \sim n^2)$.

(b) Define a function $g(n)$ such that $g = O(n^2)$, $g \neq \Theta(n^2)$ and $g \neq o(n^2)$.

Problem 3.

False Claim.

$$2^n = O(1). \tag{2}$$

Explain why the claim is false. Then identify and explain the mistake in the following bogus proof.

Bogus proof. The proof by induction on n where the induction hypothesis, $P(n)$, is the assertion (2).

base case: $P(0)$ holds trivially.

inductive step: We may assume $P(n)$, so there is a constant $c > 0$ such that $2^n \leq c \cdot 1$. Therefore,

$$2^{n+1} = 2 \cdot 2^n \leq (2c) \cdot 1,$$

which implies that $2^{n+1} = O(1)$. That is, $P(n+1)$ holds, which completes the proof of the inductive step.

We conclude by induction that $2^n = O(1)$ for all n. That is, the exponential function is bounded by a constant.

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Problem 4.
Give an elementary proof (without appealing to Stirling's formula) that $\log(n!) = \Theta(n \log n)$.

Asymptotic Notations

Let f, g be functions from $\mathbb{R}$ to $\mathbb{R}$.

- f is *asymptotically equal* to g: $f(x) \sim g(x)$ iff $\lim_{x\to\infty} f(x)/g(x) = 1$.
- f is *asymptotically smaller* than g: $f(x) = o(g(x))$ iff $\lim_{x\to\infty} f(x)/g(x) = 0$.
- for f, g nonnegative, $f = O(g)$ iff $\limsup_{x\to\infty} f(x)/g(x) < \infty$, where $\limsup_{x\to\infty} h(x) ::= \lim_{x\to\infty} \text{lub}_{y\geq x} h(y)$.
  An alternative, equivalent, definition is
  $$f = O(g) \quad \text{iff} \quad \exists c, x_0 \in \mathbb{R}^+ \, \forall x \geq x_0.\ f(x) \leq cg(x).$$
- Finally, $f = \Theta(g)$ iff $f = O(g)$ AND $g = O(f)$.