


Mathematics for Computer Science
 MIT 6.042J/18.062J

Asymptotic Notation


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Closed form for $n!$

$$n! ::= 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n = \prod_{i=1}^n i$$

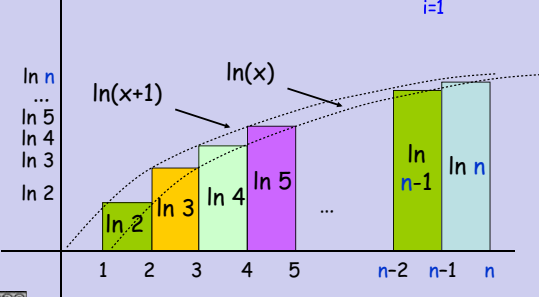
Turn product into a **sum** taking logs:


$$\begin{aligned} \ln(n!) &= \ln(1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n) = \\ &= \ln 1 + \ln 2 + \cdots + \ln(n-1) + \ln(n) \\ &= \sum_{i=1}^n \ln(i) \end{aligned}$$


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Closed form for $n!$

Integral Method to bound $\sum_{i=1}^n \ln(i)$



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Closed form for $n!$


$$n \ln\left(\frac{n}{e}\right) + 1 \leq \sum_{i=1}^n \ln(i)$$

$$\leq (n+1) \ln\left(\frac{n+1}{e}\right) + 0.6$$

reminder:

$$\int \ln x \, dx = x \ln\left(\frac{x}{e}\right)$$


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Closed form for $n!$

$$\sum_{i=1}^n \ln(i) \approx \left(n + \frac{1}{2}\right) \ln\left(\frac{n}{e}\right)$$

exponentiating:


$$n! \approx \sqrt{n/e} \left(\frac{n}{e}\right)^n$$

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Stirling's Formula

A precise approximation:

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

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Little Oh: $o(\cdot)$
 Asymptotically smaller :
 Def: $f(n) = o(g(n))$
 iff $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$

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Little Oh: $o(\cdot)$
 $n^2 = o(n^3)$
 because $\lim_{n \rightarrow \infty} \frac{n^2}{n^3} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$

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Big Oh: $O(\cdot)$
 Asymptotic Order of Growth:
 $f(n) = O(g(n))$
 $\limsup_{n \rightarrow \infty} \left(\frac{f(n)}{g(n)} \right) < \infty$
 a technicality -- ignore now

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Big Oh: $O(\cdot)$
 $3n^2 = O(n^2)$
 because $\lim_{n \rightarrow \infty} \frac{3n^2}{n^2} = 3 < \infty$

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Theta: $\Theta(\cdot)$
 Same Order of Growth:
 $f(n) = \Theta(g(n))$
 Def: $f(n) = O(g(n))$
 and
 $g(n) = O(f(n))$

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Asymptotics: Intuitive Summary

- $f \sim g$: f & g nearly equal
- $f = o(g)$: f much less than g
- $f = O(g)$: f roughly $\leq g$
- $f = \Theta(g)$: f & g roughly equal

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The Oh's

lemma:

If $f = o(g)$ or $f \sim g$, then $f = O(g)$

$\lim = 0$ or $\lim = 1$ IMPLIES $\lim < \infty$

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The Oh's

If $f = o(g)$, then $g \neq O(f)$

$\lim \frac{f}{g} = 0$ IMPLIES $\lim \frac{g}{f} = \infty$

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Big Oh: $O(\cdot)$

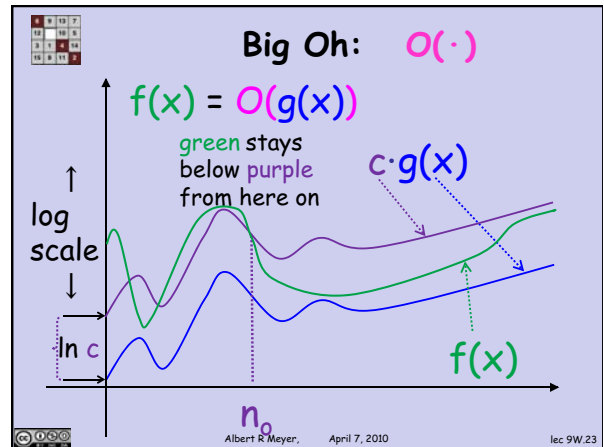
Equivalent definition:

$f(n) = O(g(n))$

$\exists c, n_0 \forall n \geq n_0.$

$f(n) \leq c \cdot g(n)$

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Little Oh: $o(\cdot)$

Lemma: $x^a = o(x^b)$ for $a < b$

Proof: $\frac{x^a}{x^b} = \frac{1}{x^{b-a}}$ and $b - a > 0$

so as $x \rightarrow \infty$, $\frac{1}{x^{b-a}} \rightarrow 0$

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Little Oh: $o(\cdot)$

Lemma: $\ln x = o(x^\epsilon)$

for $\epsilon > 0$.

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Little Oh: $o(\cdot)$

Lemma:

$$x^n = o(a^x)$$

for $a > 1$.



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Little Oh: $o(\cdot)$

proofs:

L'Hopital's Rule,
McLaurin Series
(see a Calculus text)



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Big Oh **Mistakes**

" $\cdot = O(\cdot)$ " defines a relation

Don't write $O(g) = f$.

Otherwise: $x = O(x)$, so $O(x) = x$.

But $2x = O(x)$, so

$$2x = O(x) = x,$$

therefore $2x = x$.

Nonsense!



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Team Problems

Problems

1-4



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