Solutions to In-Class Problems Week 9, Mon.

Problem 1.
An explorer is trying to reach the Holy Grail, which she believes is located in a desert shrine d days walk from the nearest oasis. In the desert heat, the explorer must drink continuously. She can carry at most 1 gallon of water, which is enough for 1 day. However, she is free to create water caches out in the desert.

For example, if the shrine were $2/3$ of a day's walk into the desert, then she could recover the Holy Grail with the following strategy. She leaves the oasis with 1 gallon of water, travels $1/3$ day into the desert, caches $1/3$ gallon, and then walks back to the oasis— arriving just as her water supply runs out. Then she picks up another gallon of water at the oasis, walks $1/3$ day into the desert, tops off her water supply by taking the $1/3$ gallon in her cache, walks the remaining $1/3$ day to the shine, grabs the Holy Grail, and then walks for $2/3$ of a day back to the oasis— again arriving with no water to spare.

But what if the shrine were located farther away?

(a) What is the most distant point that the explorer can reach and then return to the oasis if she takes only 1 gallon from the oasis?

Solution. At best she can walk $1/2$ day into the desert and then walk back. ■

(b) What is the most distant point the explorer can reach and still return to the oasis if she takes only 2 gallons from the oasis? No proof is required; just do the best you can.

Solution. The explorer walks $1/4$ day into the desert, drops $1/2$ gallon, then walks home. Next, she walks $1/4$ day into the desert, picks up $1/4$ gallon from her cache, walks an additional $1/2$ day out and back, then picks up another $1/4$ gallon from her cache and walks home. Thus, her maximum distance from the oasis is $3/4$ of a day's walk. ■

(c) The explorer will travel using a recursive strategy to go far into the desert and back drawing a total of n gallons of water from the oasis. Her strategy is to build up a cache of $n - 1$ gallons, plus enough to get home, a certain fraction of a day's distance into the desert. On the last delivery to the cache, instead of returning home, she proceeds recursively with her $n - 1$ gallon strategy to go farther into the desert and return to the cache. At this point, the cache has just enough water left to get her home.

Prove that with n gallons of water, this strategy will get her $H_n/2$ days into the desert and back, where H_n is the nth Harmonic number:

$$H_n ::= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Conclude that she can reach the shrine, however far it is from the oasis.

Solution. To build up the first cache of $n-1$ gallons, she should make n trips $1/(2n)$ days into the desert, dropping off $(n-1)/n$ gallons each time. Before she leaves the cache for the last time, she has $n-1$ gallons plus enough for the walk home. Then she applies her $(n-1)$-day strategy. So letting D_n be her maximum distance into the desert and back, we have

$$D_n = \frac{1}{2n} + D_{n-1}.$$

So

$$\begin{aligned} D_n &= \frac{1}{2n} + \frac{1}{2(n-1)} + \frac{1}{2(n-2)} + \cdots + \frac{1}{2\cdot 2} + \frac{1}{2\cdot 1} \\ &= \frac{1}{2}\left(\frac{1}{n} + \frac{1}{(n-1)} + \frac{1}{(n-2)} + \cdots + \frac{1}{\cdot 2} + \frac{1}{\cdot 1}\right) \\ &= \frac{H_n}{2}. \end{aligned}$$

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(d) Suppose that the shrine is $d = 10$ days walk into the desert. Use the asymptotic approximation $H_n \sim \ln n$ to show that it will take more than a million years for the explorer to recover the Holy Grail.

Solution. She obtains the Grail when:

$$\frac{H_n}{2} \approx \frac{\ln n}{2} \geq 10.$$

This requires $n \geq e^{20} = 4.8 \cdot 10^8$ days $> 1.329M$ years.

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Problem 2.
There is a number a such that $\sum_{i=1}^{\infty} i^p$ converges iff $p < a$. What is the value of a? Prove it.

Solution. $a = -1$.

For $p = -1$, the sum is the harmonic series which we know does not converge. Since the term i^p is increasing in p for $i > 1$, the sum will be larger, and hence also diverge for $p > -1$.

For $p < -1$ there exists an $\epsilon > 0$ such that $p = -(1+\epsilon)$. By the integral method,

$$\begin{aligned} \sum_{i=1}^{\infty} i^{-(1+\epsilon)} &\leq 1 + \int_1^{\infty} x^{-(1+\epsilon)}\, dx \\ &= 1 + \epsilon^{-1} - \epsilon^{-1} \lim_{\alpha\to\infty} \alpha^{-\epsilon} \\ &= 1 + \epsilon^{-1} \\ &< \infty \end{aligned}$$

Hence the sum is bounded above, and since it is increasing, it has a finite limit, that is, it converges.

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Problem 3.
Suppose $f, g : \mathbb{R} \to \mathbb{R}$ and $f \sim g$.

(a) Prove that $2f \sim 2g$.

$$\frac{2f}{2g} = \frac{f}{g},$$

so they have the same limit as $n \to \infty$.

(b) Prove that $f^2 \sim g^2$.

Solution.

$$\lim_{n\to\infty} \frac{f^2}{g^2} = \lim_{n\to\infty} \frac{f}{g} \cdot \frac{f}{g} = \lim_{n\to\infty} \frac{f}{g} \cdot \lim_{n\to\infty} \frac{f}{g} = 1 \cdot 1 = 1.$$

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(c) Give examples of f and g such that $2^f \not\sim 2^g$.

Solution. Let $f(n) ::= n,\ g(n) ::= n + \log n$. Then

$$\frac{2^{f(n)}}{2^{g(n)}} = \frac{2^n}{2^{n+\log n}} = \frac{2^n}{2^n 2^{\log n}} = \frac{1}{n},$$

so

$$\lim_{n\to\infty} \frac{2^{f(n)}}{2^{g(n)}} = \lim_{n\to\infty} \frac{1}{n} = 0 \neq 1.$$

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6.042J / 18.062J Mathematics for Computer Science
Spring 2010