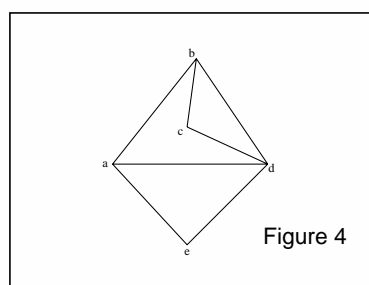
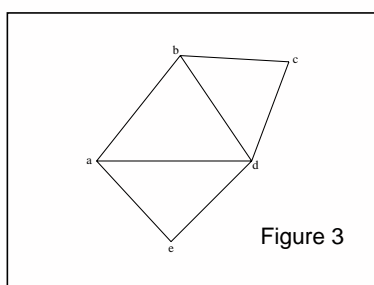
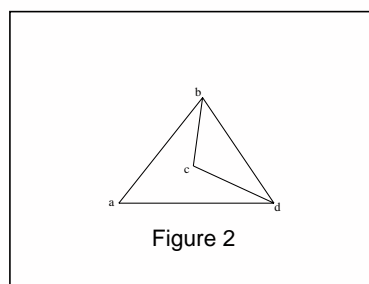
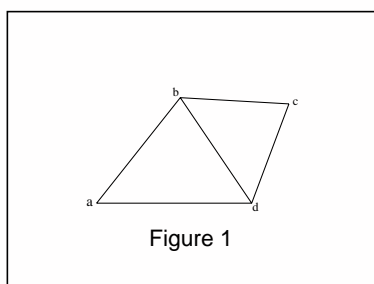


## Solutions to In-Class Problems Week 7, Fri.

### Problem 1.

Figures 1–4 show different pictures of planar graphs.



(a) For each picture, describe its discrete faces (simple cycles that define the region borders).

**Solution.** Figs 1 & 2:  $abda, bcdb, abcd$ . Fig 3:  $abcdea, adea, abda, bcdb$ . Fig 4:  $abcd, abdea, bdc, b, adea$ . ■

(b) Which of the pictured graphs are isomorphic? Which pictures represent the same *planar embedding*? – that is, they have the same discrete faces.

**Solution.** Figs 1 & 2 have the same faces, so are different pictures of the *same* planar drawing. Figs 3 & 4 both have four faces, but they are different, for example, Fig 3 has a face with 5 edges, but the longest face in Fig 4 has 4 edges. ■

(c) Describe a way to construct the embedding in Figure 4 according to the recursive Definition 12.3.1 of planar embedding. For each application of a constructor rule, be sure to indicate the faces (cycles) to which the rule was applied and the cycles which result from the application.

**Solution.** Here's one way. (By Lemma 12.7.1, the constructor steps could be done in any order.)

recursive step	faces
vertex $a$ (base case)	$a$
vertex $b$ (base)	$b$
$a-b$ (bridge)	$aba$
vertex $c$ (base)	$c$
$b-c$ (bridge)	$abcba$
vertex $d$ (base)	$d$
$c-d$ (bridge)	$abcdcba$
$a-d$ (split)	$dabcd, dabcd$
$b-d$ (split)	$dabd, dbcd, abcd$
vertex $e$ (base)	$e$
$d-e$ (bridge)	$dedabd, dbcd, abcd$
$a-e$ (split)	$abdea, adea, dbcd, abcd$

### Problem 2.

Prove the following assertions by structural induction on the definition of planar embedding.

(a) In a planar embedding of a graph, each edge is traversed a total of two times by the faces of the embedding.

**Solution.** *Proof.* The induction hypothesis is that if  $\mathcal{E}$  is a planar embedding of a graph, then each edge is traversed exactly twice by the faces of  $\mathcal{E}$ .

**Base case:** There is one vertex and no edges, so this case holds vacuously.

**Constructor case:** (face-splitting) The only change is that one face of  $\mathcal{E}$  splits into two new faces, each traversing the new edge once.

**Constructor case:** (bridge between two connected graphs) The only change is that two faces merge into one face that makes two traversals of the new bridging edge. So the traversals of other edges are unchanged, and the new edge is traversed twice by the new face.

So in any case, all edges of  $\mathcal{E}$  are traversed exactly twice. This completes the proof of the Constructor case. We conclude by structural induction that for all planar embeddings,  $\mathcal{E}$ , then each edge is traversed exactly twice by the faces of  $\mathcal{E}$ . ■

(b) In a planar embedding of a connected graph with at least three vertices, each face is of length at least three.

**Solution.** *Proof.* The induction hypothesis is that if  $\mathcal{E}$  is a planar embedding of a graph with at least three vertices, then all faces in  $\mathcal{E}$  are of length at least three.

**Base case:** There is one vertex, so this case holds vacuously.

**Constructor case:** (face-splitting) An edge  $a-b$  is added between nonadjacent vertices  $a, b$  on the same face. This face is replaced by two new faces of the form  $abc \dots a$  and  $abd \dots a$  where  $c \neq d$  are vertices different from  $a$  and  $b$ . So both new faces are of length at least 3; no other faces change.

**Constructor case:** (bridge between two connected graphs)

**case 1:** (both graphs have one vertex). Connecting these graphs with a bridge gives a graph with fewer than three vertices, so this case holds vacuously.

**case 2:** (one graph has exactly two vertices and the other has at most two vertices). Connecting these graphs with a bridge yields a line graph of length two or three whose unique embedding is a cycle of length four or six going from one end of the graph to the other and back. In any case, the one face has length more than three.

**case 3:** (one graph has at most two vertices and the other has at least three vertices). Connecting replaces the face of the vertex graph with at most two vertices and a face of the other graph with a face of length at least  $2 + 3 = 5$ , and leaves all other faces unchanged. So all faces are indeed of length at least three.

**case 4:** (both graphs have at least three vertices). Connecting replaces two faces of length at least three by a single face of length at least  $2 + 3 + 3 = 7$ , and leaves all other faces unchanged. So all faces are indeed of length at least three.

So in any case, all faces of connected planar embedding of graphs with at least three vertices are indeed of length at least three. This completes the proof of the Constructor case and the structural induction. ■

**Problem 3. (a)** Show that if a connected planar graph with more than two vertices is bipartite, then

$$e \leq 2v - 4. \quad (1)$$

*Hint:* Similar to the proof of Corollary 12.6.3 that for planar graphs  $e \leq 3v - 6$ .

**Solution.** By Problem 2.b, every face is of length at least 3. But in a bipartite graph there are no cycles of odd length, so all must be of length at least 4.

Each edge is traversed by exactly two faces, so

$$2e = \sum_{f \in \text{faces}} \text{length}(f) \geq \sum_{f \in \text{faces}} 4 = 4f. \quad (2)$$

By Euler's formula,  $f = e - v + 2$ , so substituting for  $f$  in (2), yields

$$2e \geq 4(e - v + 2),$$

which simplifies to (1). ■

(b) Conclude that that  $K_{3,3}$  is not planar. ( $K_{3,3}$  is the graph with six vertices and an edge from each of the first three vertices to each of the last three.)

**Solution.**  $K_{3,3}$  is bipartite and connected. Also, it has 9 edges and 6 vertices, and since  $9 > 8 = 2 \cdot 6 - 4$ , it does not satisfy (1), and so cannot be planar. ■

## Appendix

**Definition 3.1.** A *planar embedding* of a *connected* graph consists of a nonempty set of cycles of the graph called the *discrete faces* of the embedding. Planar embeddings are defined recursively as follows:

- **Base case:** If  $G$  is a graph consisting of a single vertex,  $v$ , then a planar embedding of  $G$  has one discrete face, namely the length zero cycle,  $v$ .
- **Constructor Case:** (split a face) Suppose  $G$  is a connected graph with a planar embedding, and suppose  $a$  and  $b$  are distinct, nonadjacent vertices of  $G$  that appear on some discrete face,  $\gamma$ , of the planar embedding. That is,  $\gamma$  is a cycle of the form

$$a \dots b \dots a.$$

Then the graph obtained by adding the edge  $a-b$  to the edges of  $G$  has a planar embedding with the same discrete faces as  $G$ , except that face  $\gamma$  is replaced by the two discrete faces<sup>1</sup>

$$a \dots ba \quad \text{and} \quad ab \dots a,$$

as illustrated in Figure 1.

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<sup>1</sup> There is one exception to this rule. If  $G$  is a line graph beginning with  $a$  and ending with  $b$ , then the cycles into which  $\gamma$  splits are actually the same. That's because adding edge  $a-b$  creates a simple cycle graph,  $C_n$ , that divides the plane into an "inner" and an "outer" region with the same border. In order to maintain the correspondence between continuous faces and discrete faces, we have to allow two "copies" of this same cycle to count as discrete faces. But since this is the only situation in which two faces are actually the same cycle, this exception is better explained in a footnote than mentioned explicitly in the definition.

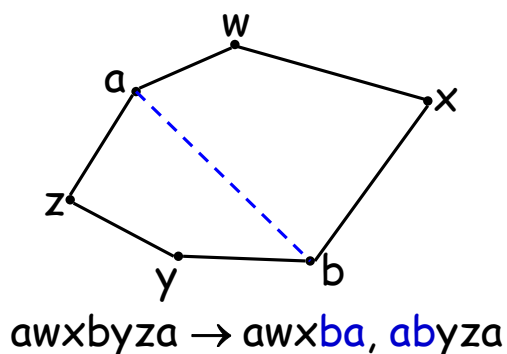


Figure 1: The Split a Face Case.

- **Constructor Case:** (add a bridge) Suppose  $G$  and  $H$  are connected graphs with planar embeddings and disjoint sets of vertices. Let  $a$  be a vertex on a discrete face,  $\gamma$ , in the embedding of  $G$ . That is,  $\gamma$  is of the form

$$a \dots a.$$

Similarly, let  $b$  be a vertex on a discrete face,  $\delta$ , in the embedding of  $H$ , so  $\delta$  is of the form

$$b \dots b.$$

Then the graph obtained by connecting  $G$  and  $H$  with a new edge,  $a-b$ , has a planar embedding whose discrete faces are the union of the discrete faces of  $G$  and  $H$ , except that faces  $\gamma$  and  $\delta$  are replaced by one new face

$$a \dots ab \dots ba,$$

as illustrated in Figure 2.

An arbitrary graph is *planar* iff each of its connected components has a planar embedding.

**Theorem 3.2** (Euler's Formula). *If a connected graph has a planar embedding, then*

$$v - e + f = 2$$

where  $v$  is the number of vertices,  $e$  is the number of edges, and  $f$  is the number of faces.

**Corollary 3.3.** *Suppose a connected planar graph has  $v \geq 3$  vertices and  $e$  edges. Then*

$$e \leq 3v - 6.$$

*Proof.* By definition, a connected graph is planar iff it has a planar embedding. So suppose a connected graph with  $v$  vertices and  $e$  edges has a planar embedding with  $f$  faces. By Problem 2.a, every edge is traversed exactly twice by the face boundaries. So the sum of the lengths of the face

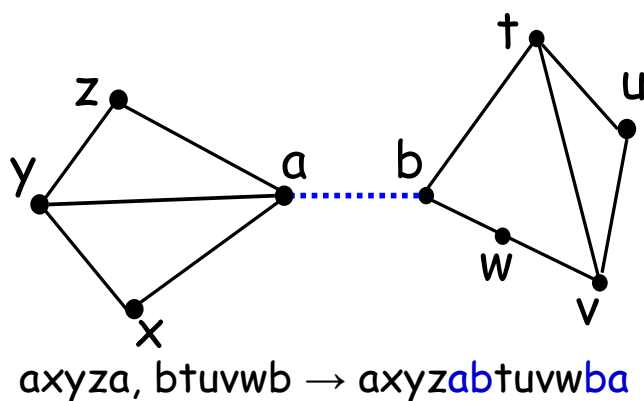


Figure 2: The Add Bridge Case.

boundaries is exactly  $2e$ . Also by Problem 2.b, when  $v \geq 3$ , each face boundary is of length at least three, so this sum is at least  $3f$ . This implies that

$$3f \leq 2e. \quad (3)$$

But  $f = e - v + 2$  by Euler's formula, and substituting into (3) gives

$$\begin{aligned} 3(e - v + 2) &\leq 2e \\ e - 3v + 6 &\leq 0 \\ e &\leq 3v - 6 \end{aligned}$$

■

**Corollary 3.4.**  $K_5$  is not planar.

*Proof.*

$$e = 10 > 9 = 3v - 6.$$

■

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