

6	9	13	7
12	10	5	
3	1	14	11
15	8	16	2

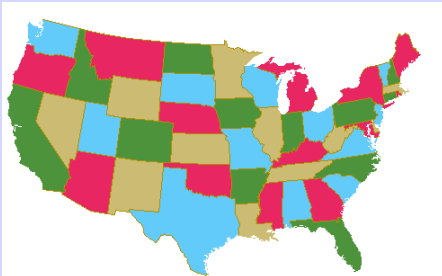
Mathematics for Computer Science
MIT 6.042J/18.062J

Planar Graphs

Albert R Meyer, March 19, 2010 lec 7F.1

6	9	13	7
12	10	5	
3	1	14	11
15	8	16	2

Planar Graphs



Albert R Meyer, March 19, 2010 lec 7F.2

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6	9	13	7
12	10	5	
3	1	14	11
15	8	16	2

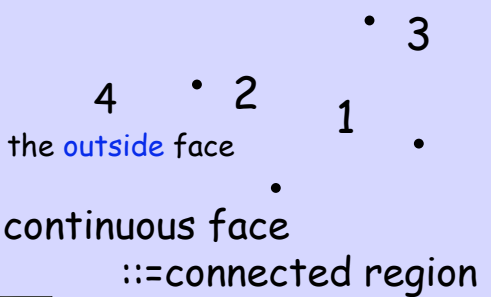
Planar Graphs

A graph is **planar** if there is a way to **draw** it in the plane without edges crossing.

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6	9	13	7
12	10	5	
3	1	14	11
15	8	16	2

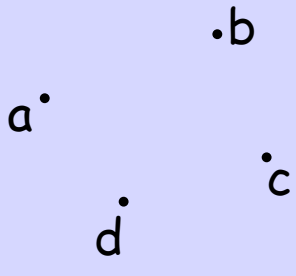
4 Continuous Faces here



Albert R Meyer, March 19, 2010 lec 7F.4

6	9	13	7
12	10	5	
3	1	14	11
15	8	16	2

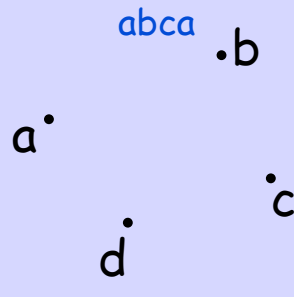
Region Boundaries



Albert R Meyer, March 19, 2010 lec 7F.5

6	9	13	7
12	10	5	
3	1	14	11
15	8	16	2

Region Boundaries



Albert R Meyer, March 19, 2010 lec 7F.6

6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Region Boundaries

abca
 abda

Albert R Meyer, March 19, 2010 lec 7F.7

6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Region Boundaries

abca
 abda
 bcdb

Albert R Meyer, March 19, 2010 lec 7F.9

6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Region Boundaries

abca
 abda
 bcdb
 acda

Albert R Meyer, March 19, 2010 lec 7F.10

6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Region Boundaries: Bridge

abcda
 efge
 abcdefgceda

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6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Region Boundaries: Dongle

rstur
 stvxyxvwvt

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6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Region Boundaries: Dongle

stvxyxvwvt
 rstur

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6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

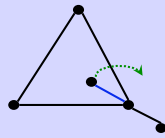
Planar Embedding

A **planar embedding** is a connected graph *along with* its face boundary cycles (same graph may have different embeddings)

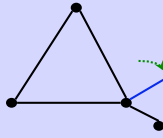
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6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Same graph, different embeddings



2 length 5 faces



length 3 face
length 7 face

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6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Recursive Def: Planar Embeddings

Base: a graph consisting of

- single vertex, v ,
- with a single face:
length 0 cycle from v to v , is a **PE**.

v
●
graph

v
face

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6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Adding an edge to an embedding

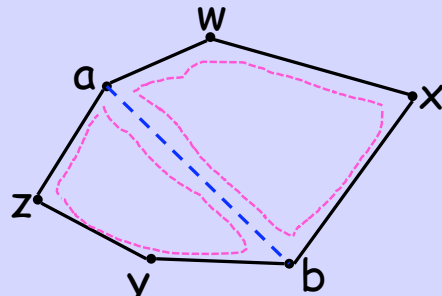
Two constructor cases:

- 1) Add edge across a face (splits face in two)
- 2) Add bridge between connected components (merges 2 outer faces)

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6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Constructor: Split a Face

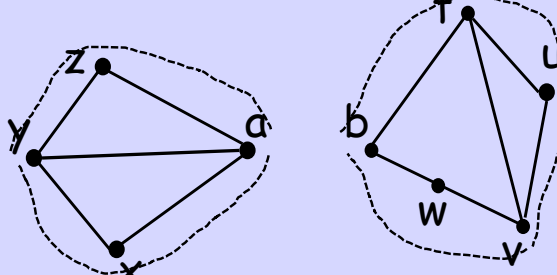


$awxbyza \rightarrow awxba, abyza$

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6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Constructor: Add a Bridge



$axyza \quad btuvw$

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Constructor: Add a Bridge

axyza, btuvwb

Albert R Meyer, March 19, 2010 lec 7F.23

Constructor: Add a Bridge

axyza, btuvwb \rightarrow axyzabtuvwba

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Team Problem

Problem 1

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Euler's Formula

If a planar embedding has v vertices, e edges, and f faces, then

$$v - e + f = 2$$

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Euler's Formula

Proof by structural induction on embeddings:

base case: 1 vertex

$$v = 1, e = 0, f = 1$$

$$1 - 0 + 1 = 2$$

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Adding an edge to a drawing

Constructor case (split face):

- v stays the same
- e increases by 1
- f increases by 1

so $v - e + f$ stays the same

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6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2


Adding an edge to a drawing
Constructor case (add bridge):

$$v = v_1 + v_2$$

$$-e = -(e_1 + e_2 + 1)$$

$$f = f_1 + f_2 - 1 \quad (\text{two outer faces merge into one})$$

$$2 = 2 + 2 - 2$$



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6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Planar Properties

- an edge appears twice on faces
- face length ≥ 3 (for $v \geq 3$)

$$3(e-v+2) = 3f \leq 2e$$

combining with Euler

$$e \leq 3v-6$$

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6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Planar Properties

$$e \leq 3v-6$$

Cor: K_5 is not planar

pf: $v = 5, e = 10$
 $10 \not\leq 3 \cdot 5 - 6$

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6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Planar Properties

$$e \leq 3v-6$$

Cor: Every planar graph has a vertex of degree ≤ 5

pf: suppose all degrees ≥ 6

Then

$$6v \leq \sum \text{degrees} = 2e \leq 6v-12$$

contradiction!

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6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Planar Properties

Cor: Every planar graph has a vertex of degree ≤ 5

Therefore,

every planar graph is 6-colorable

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6	9	13	7
12	10	5	
3	1	14	11
15	8	11	2

Euler's Formula

Cor: There are at most 5 regular polyhedra (proof in Notes)

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6	9	13	7
12	10	5	
3	1	14	11
15	8	4	2

Team Problems

Problems

2 & 3



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Spring 2010

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