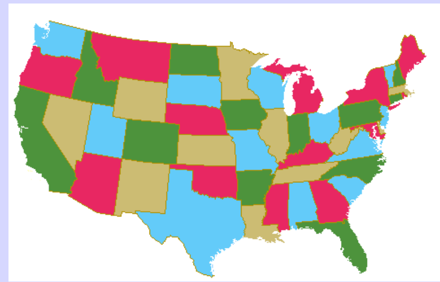
Mathematics for Computer Science
MIT 6.042J/18.062J

Planar Graphs

Planar Graphs



Planar Graphs

A graph is planar if there is a way to draw it in the plane without edges crossing.

4 Continuous Faces here

3

4 2

1

the outside face

continuous face
::=connected region

Region Boundaries

b

a

c

d

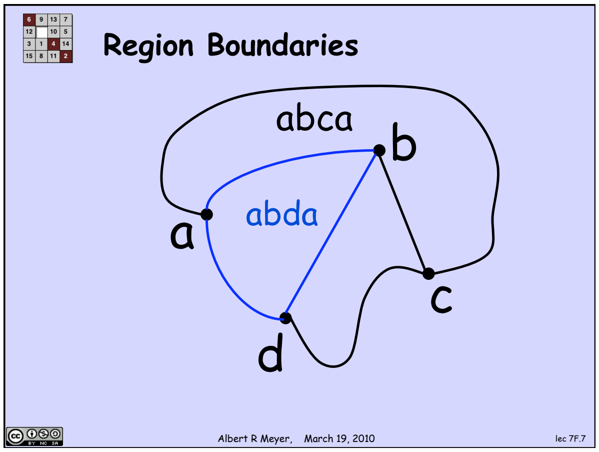
Region Boundaries

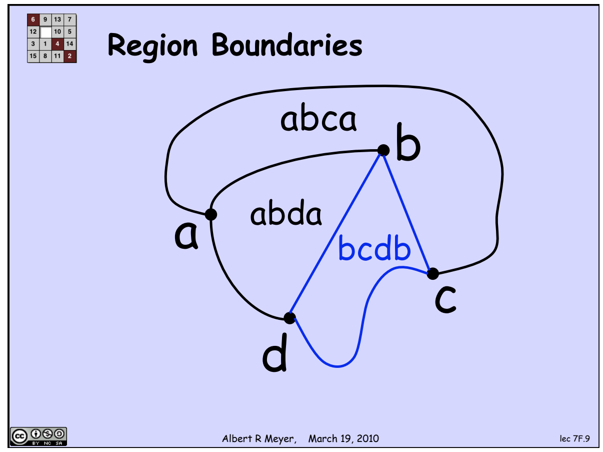
abca b

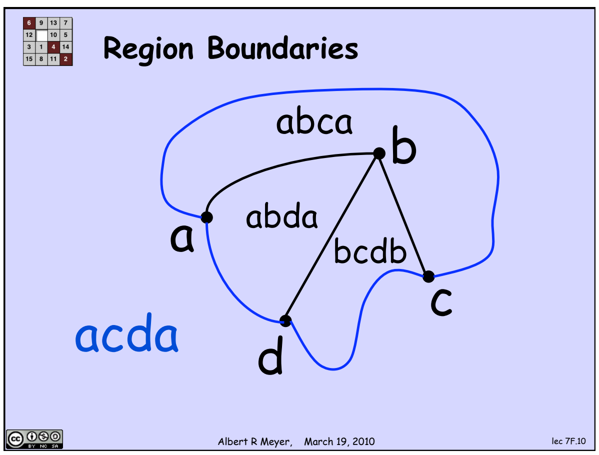
a

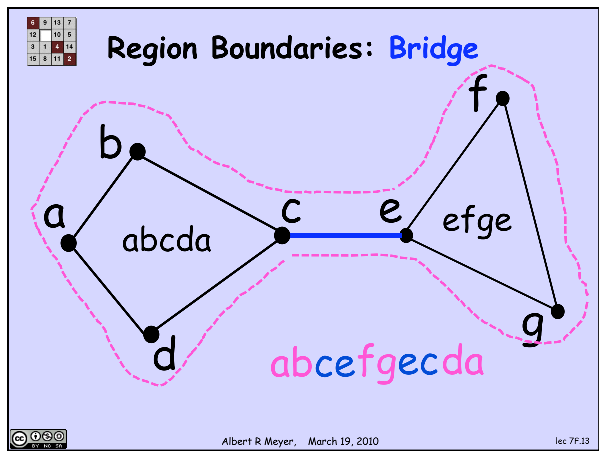
c

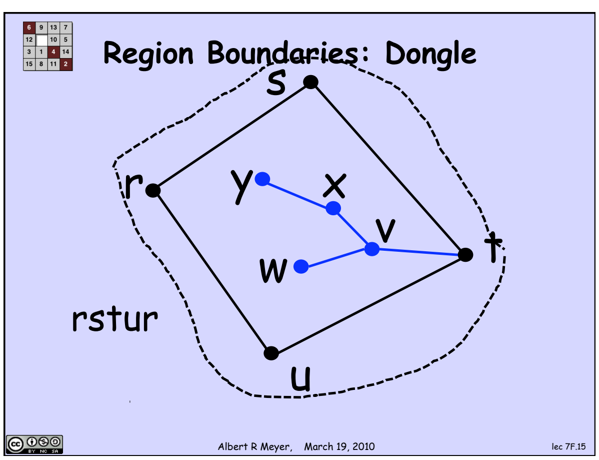
d

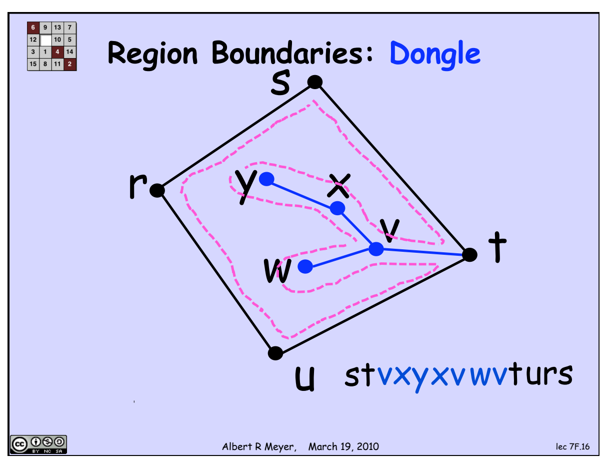

Region Boundaries
abca
abda
a
b
c
d
Albert R Meyer, March 19, 2010
lec 7F.7


Region Boundaries
abca
abda
bcdb
a
b
c
d
Albert R Meyer, March 19, 2010
lec 7F.9


Region Boundaries
abca
abda
bcdb
acda
a
b
c
d
Albert R Meyer, March 19, 2010
lec 7F.10


Region Boundaries: Bridge
abcda
efge
abcefgecda
a
b
c
d
e
f
g
Albert R Meyer, March 19, 2010
lec 7F.13

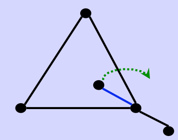

Region Boundaries: Dongle
rstur
s
r
y
x
v
w
t
u
Albert R Meyer, March 19, 2010
lec 7F.15


Region Boundaries: Dongle
stvxyxvwvturs
s
r
y
x
v
w
t
u
Albert R Meyer, March 19, 2010
lec 7F.16

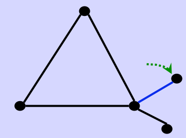
Planar Embedding

A planar embedding is a connected graph *along with* its face boundary cycles (same graph may have different embeddings)

Same graph, different embeddings



2 length 5 faces



length 3 face
length 7 face

Recursive Def: Planar Embeddings

Base: a graph consisting of
- single vertex, v,
- with a single face:
  length 0 cycle from v to v,

is a PE.

v ● graph

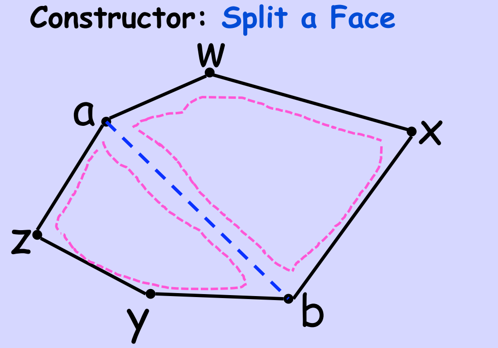
v face

Adding an edge to an embedding

Two constructor cases:

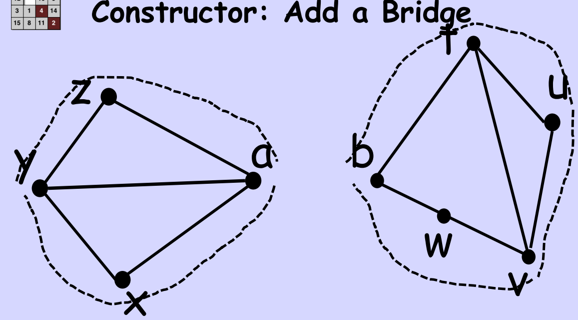
1) Add edge across a face (splits face in two)
2) Add bridge between connected components (merges 2 outer faces)

Constructor: Split a Face



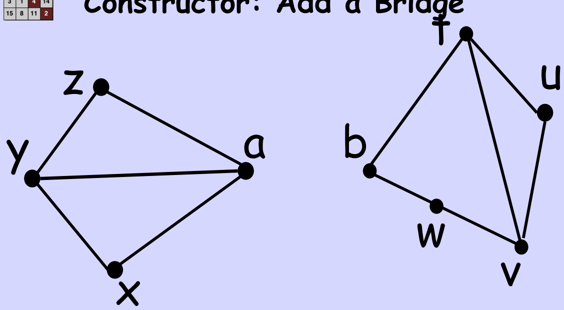
awxbyza → awxba, abyza

Constructor: Add a Bridge



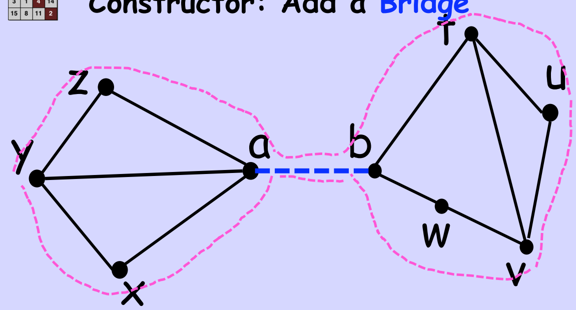
axyza btuvwb

Constructor: Add a Bridge



axyza, btuvwb

Constructor: Add a Bridge



axyza, btuvwb → axyzabtuvwba

Team Problem

Problem 1

Euler's Formula

If a planar embedding has v vertices, e edges, and f faces, then

$$v - e + f = 2$$

Euler's Formula

Proof by structural induction on embeddings:

base case: 1 vertex

$$v = 1, e = 0, f = 1$$

$$1 - 0 + 1 = 2$$

O.K.

Adding an edge to a drawing

Constructor case (split face):

v stays the same

e increases by 1

f increases by 1

so v - e + f stays the same

O.K.

Adding an edge to a drawing

Constructor case (add bridge):

$$
\begin{aligned}
v &= v_1 + v_2 \\
-e &= -(e_1 + e_2 + 1) \\
f &= f_1 + f_2 - 1 \quad \text{(two outer faces merge into one)} \\
\hline
2 &= 2 + 2 - 2
\end{aligned}
$$

Planar Properties

- an edge appears twice on faces
- face length ≥ 3 (for $v \geq 3$)

$$3(e-v+2) = 3f \leq 2e$$

combining with Euler

$$e \leq 3v-6$$

Planar Properties

$$e \leq 3v-6$$

Cor: K_5 is not planar

pf: $v = 5$, $e = 10$

$$10 \not\leq 3 \cdot 5 - 6$$

Planar Properties

$$e \leq 3v-6$$

Cor: Every planar graph has a vertex of degree ≤ 5

pf: suppose all degrees ≥ 6

Then

$$6v \leq \sum \text{degrees} = 2e \leq 6v-12$$

contradiction!

Planar Properties

Cor: Every planar graph has a vertex of degree ≤ 5

Therefore,

every planar graph is 6-colorable

Euler's Formula

Cor: There are at most 5 regular polyhedra (proof in Notes)

Team Problems

Problems 2 & 3