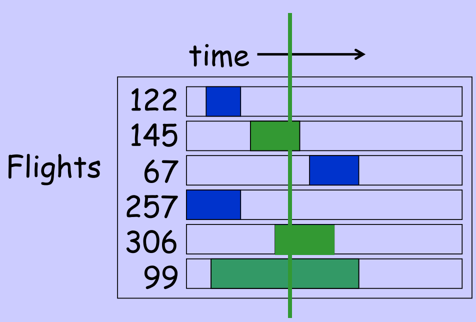
Mathematics for Computer Science
MIT 6.042J/18.062J

Graph Coloring
Bipartite Matching

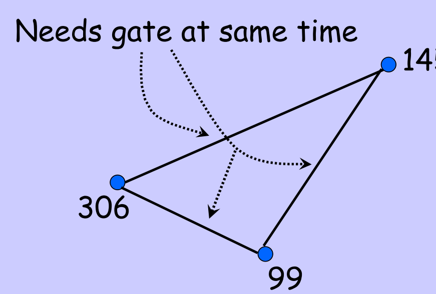
Flight Gates

flights need gates, but times overlap.
how many gates needed?

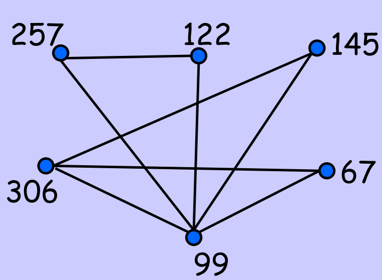
Airline Schedule



Conflicts Among 3 Flights



Model all Conflicts with a Graph

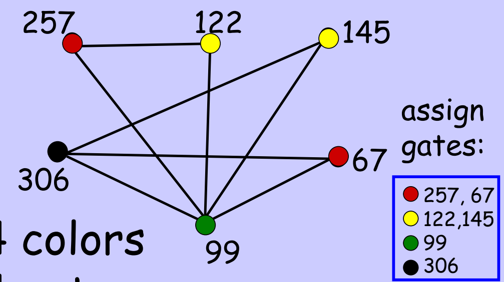


Color the vertices

Color the vertices so that adjacent vertices have different colors.

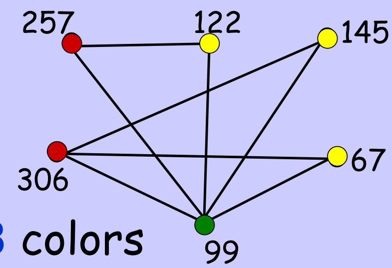
min # distinct colors needed =
min # gates needed

Coloring the Vertices



4 colors
4 gates

Better coloring



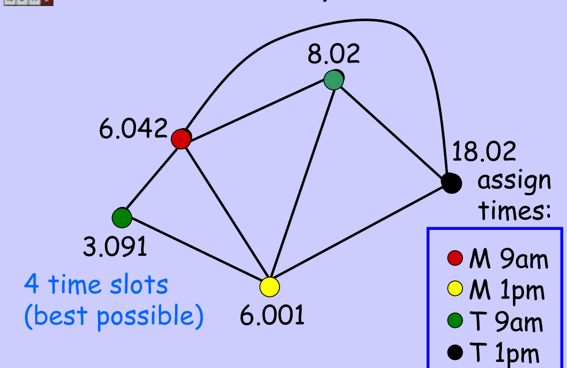
3 colors
3 gates

Final Exams

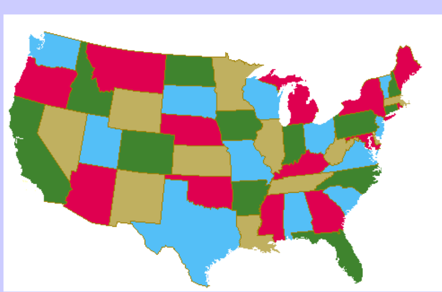
subjects conflict if student takes both, so need different time slots.

how short an exam period?

Model as a Graph



Map Coloring



Planar Four Coloring

any planar map is 4-colorable.
1850's: false proof published
(was correct for 5 colors).
1970's: proof with computer
1990's: much improved

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Chromatic Number

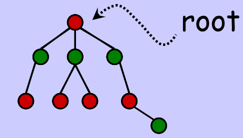
min #colors for G is

chromatic number, $\chi(G)$

lemma:

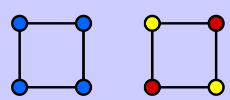
$$\chi(\text{tree}) = 2$$

Trees are 2-colorable

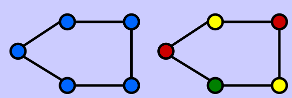


Pick any vertex as "root."
if (unique) path from root is
even length: red
odd length: green

Simple Cycles

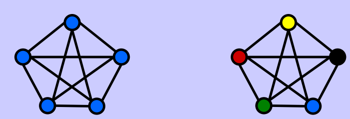


$\chi(C_{even}) = 2$



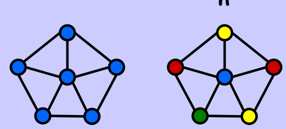
$\chi(C_{odd}) = 3$

Complete Graph K_5



$$\chi(K_n) = n$$

The Wheel W_n



W_5

$\chi(W_{odd}) = 4$

$\chi(W_{even}) = 3$

Bounded Degree

all degrees $\leq k$, implies

$$\chi(G) \leq k+1$$

very simple algorithm…

"Greedy" Coloring

…color vertices in any order.
next vertex gets a color different from its neighbors.

$\leq$ k neighbors, so
k+1 colors always work

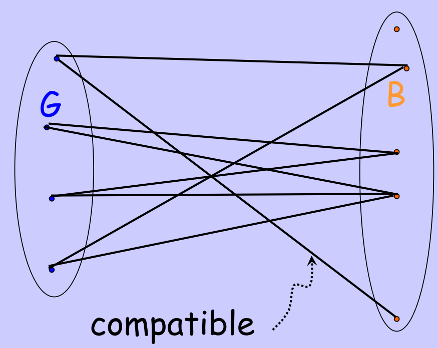
coloring arbitrary graphs

2-colorable? --easy to check
3-colorable? --hard to check
(even if planar)

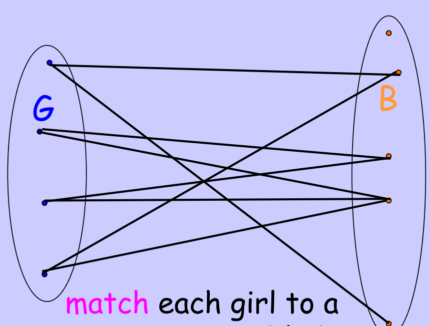
find $\chi(G)$? --theoretically no harder than 3-color, but harder in practice

Bipartite Matching

Compatible Boys & Girls

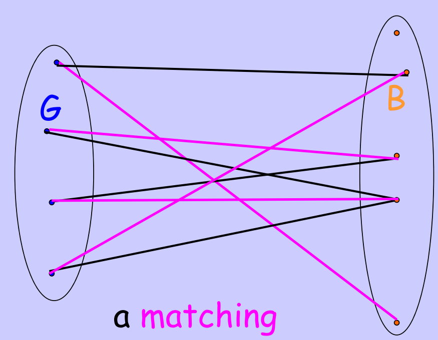


Compatible Boys & Girls

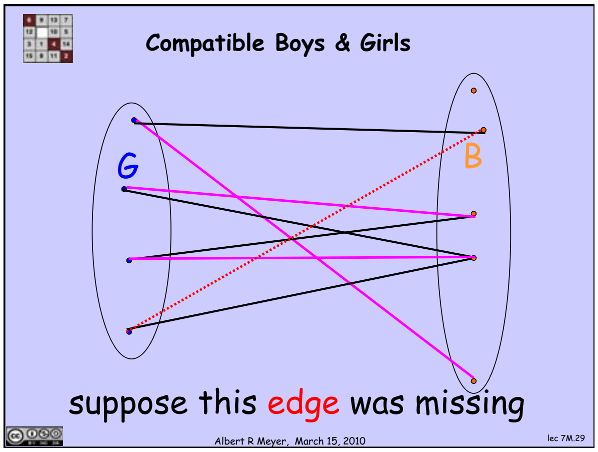


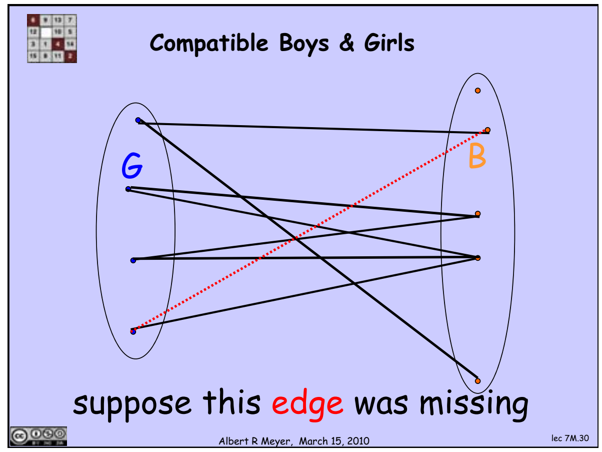
match each girl to a unique compatible boy

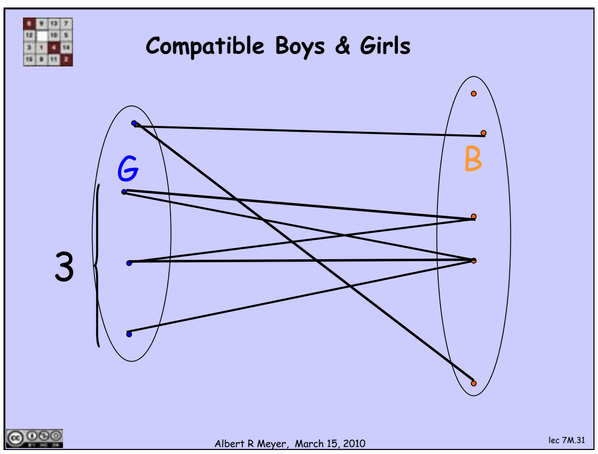
Compatible Boys & Girls

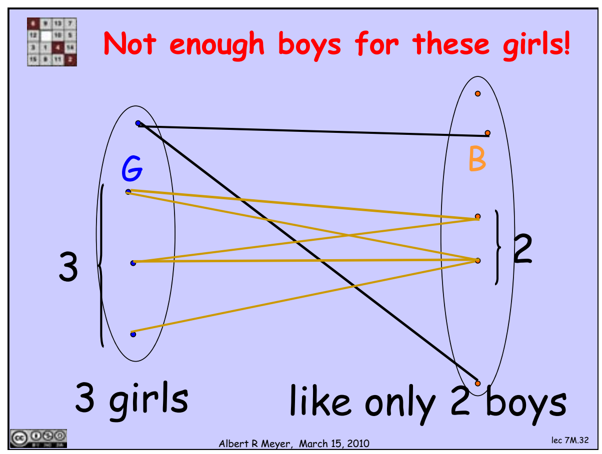


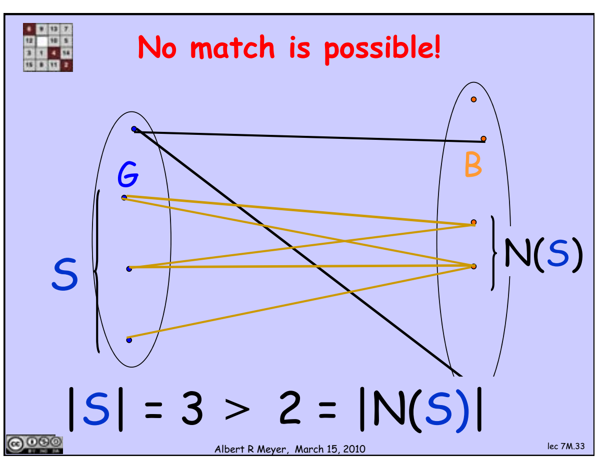
a matching

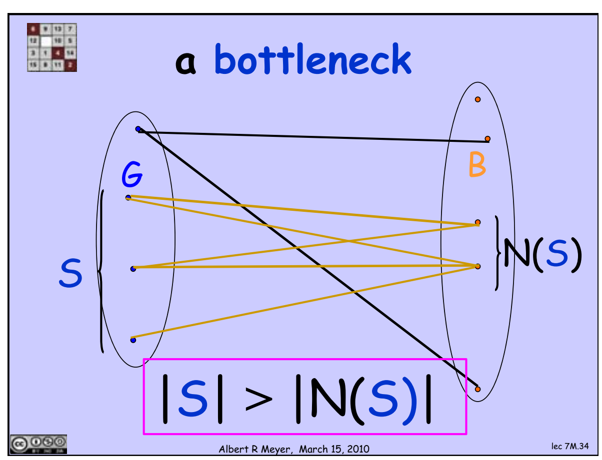

Compatible Boys & Girls
G
B
suppose this edge was missing
Albert R Meyer, March 15, 2010
lec 7M.29


Compatible Boys & Girls
G
B
suppose this edge was missing
Albert R Meyer, March 15, 2010
lec 7M.30


Compatible Boys & Girls
G
B
3
Albert R Meyer, March 15, 2010
lec 7M.31


Not enough boys for these girls!
G
B
3
2
3 girls
like only 2 boys
Albert R Meyer, March 15, 2010
lec 7M.32


No match is possible!
G
B
S
N(S)
|S| = 3 > 2 = |N(S)|
Albert R Meyer, March 15, 2010
lec 7M.33


a bottleneck
G
B
S
N(S)
|S| > |N(S)|
Albert R Meyer, March 15, 2010
lec 7M.34

Bottleneck Lemma

If there is a bottleneck, then no match is possible, obviously.

Hall's Theorem

Conversely, if there are no bottlenecks, then there is a match.

Hall's Theorem

Hall's condition

If $|S| \leq |N(S)|$ for all sets of girls, S, then there is a match.

(proof in Notes)

How to verify no bottlenecks?

fairly efficient matching procedure is known (explained in algorithms subjects)

...but there is a special situation which ensures a match:

How to verify no bottlenecks?

If every girl likes $\geq$ d boys, and every boy likes $\leq$ d girls, then no bottlenecks.

proof: say set S of girls has e incident edges:

$$\not{d} \cdot |S| \leq e \leq \not{d} \cdot |N(S)|$$

$$|S| \leq |N(S)|$$

so no bottleneck

How to verify no bottlenecks?

If every girl likes $\geq$ d boys, and every boy likes $\leq$ d girls, then no bottlenecks.

a degree-constrained bipartite graph

Team Problems

Problems 1—4

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