Solutions to In-Class Problems Week 6, Fri.

Problem 1.
Prove that a graph is a tree iff it has a unique simple path between any two vertices.

Solution. Theorem 10.3.1 shows that in a tree there are unique simple paths between any two vertices.

Conversely, suppose we have a graph, G, with unique paths. Now G is connected since there is a path between any two vertices. So we need only show that G has no simple cycles. But if there was a simple cycle in G, there are two paths between any two vertices on the cycle (going one way around the cycle or the other way around), a violation of uniqueness. So G must cannot have any simple cycles. ■

Problem 2.
The n-dimensional hypercube, H_n, is a graph whose vertices are the binary strings of length n. Two vertices are adjacent if and only if they differ in exactly 1 bit. For example, in H_3, vertices `111` and `011` are adjacent because they differ only in the first bit, while vertices `101` and `011` are not adjacent because they differ at both the first and second bits.

(a) Prove that it is impossible to find two spanning trees of H_3 that do not share some edge.

Solution. H_3 has 8 vertices so every spanning tree has 7 edges. But H_3 has only 12 edges, so any two sets of 7 edges must overlap.

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(b) Verify that for any two vertices $x \neq y$ of H_3, there are 3 paths from x to y in H_3, such that, besides x and y, no two of those paths have a vertex in common.

Solution. Define the distance between two binary strings of length n to be the number of positions at which they differ (this is known as the *Hamming distance* between the strings).

To show that there are 3 paths between any two distance 1 strings, we can, by symmetry, just consider paths between the vertices `000` and `001`.

Paths from `000` to `001`:

$$\texttt{000}, \texttt{001}$$
$$\texttt{000}, \texttt{010}, \texttt{011}, \texttt{001}$$
$$\texttt{000}, \texttt{100}, \texttt{101}, \texttt{001}$$

Likewise for distance 2, it is enough to find paths between `000` and `011`:

$$\begin{aligned} &\texttt{000}, \texttt{010}, \texttt{011} \\ &\texttt{000}, \texttt{001}, \texttt{011} \\ &\texttt{000}, \texttt{100}, \texttt{110}, \texttt{111}, \texttt{011} \end{aligned}$$

Finally, for distance 3 from `000` to `111`:

$$\begin{aligned} &\texttt{000}, \texttt{001}, \texttt{011}, \texttt{111} \\ &\texttt{000}, \texttt{010}, \texttt{110}, \texttt{111} \\ &\texttt{000}, \texttt{100}, \texttt{101}, \texttt{111} \end{aligned}$$

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(c) Conclude that the connectivity of H_3 is 3.

Solution. Since there are three paths from x to y in H_3 that share no edges with one another, removing any two edges will leave one of these paths intact, so x and y remain connected. So removing two edges from H_3 does not disconnect it.

On the other hand, removing all 3 edges incident to any vertex, disconnects that vertex. Thus the minimum number of edges necessary to disconnect H_3 is 3. ■

(d) Try extending your reasoning to H_4. (In fact, the connectivity of H_n is n for all $n \geq 1$. A proof appears in the problem solution.)

Solution. Two paths in a graph are said to *cross* when they have a vertex in common other than their endpoints. A set of paths in a graph *don't cross* when no two paths in the set cross. A graph is *k-routed* if between every pair of distinct vertices in the graph there is a set of k paths that don't cross.

We'll show that

Lemma 2.1.

$$H_n \textit{ is } n\textit{-routed for all } n \geq 1.$$

Since H_n can be disconnected by deleting the n edges incident to any vertex, this implies that H_n has connectivity n.

Proof. The proof is by induction on n with induction hypothesis,

$$P(n) ::= H_n \text{ is } n\text{-routed.}$$

Base case $[n = 1]$: Since H_1 consists of two vertices connected by an edge, $P(1)$ is immediate.

Base case $[n = 2]$: H_2 is a square. Vertices on opposite corners are obviously connected by two length 2 paths that don't cross, and adjacent vertices are connected by a length 1 path and a length 3 path.

Inductive step: We prove $P(n + 1)$ for $n \geq 2$ by letting v and w be two vertices of H_{n+1} and describing $n + 1$ paths between them that don't cross.

Let R by any positive length path in H_n, say

$$R = r_0, r_1, \ldots, r_k.$$

For $b \in \{0, 1\}$ define the H_{n+1} path

$$bR ::= br_0, br_1, \ldots, br_k.$$

Case 1: The distance from v to w is $d \leq n$. In this case, the $(n + 1)$-bit strings v and w agree in one or more positions. By symmetry, we can assume without loss of generality that v and w both start with 0. That is $v = 0v'$ and $w = 0w'$ for some n-bit strings v', w'. Now by induction, there are paths, Q_i for $1 \leq i \leq n$, that don't cross going between v' and w' in H_n.

Define the first n paths in H_{n+1} between v and w to be

$$\pi_i ::= 0Q_i$$

for $1 \leq i \leq n$. These paths don't cross since the Q_i's don't cross.

Then define the $n + 1$st path

$$\pi_{n+1} ::= v,\ 1\pi_{v',w'},\ w$$

where $\pi_{v',w'}$ is any simple path from v' to w' in H_n. Then π_{n+1} obviously does not cross any of the other paths since $1\pi_{v',w'}$ is vertex disjoint from $0Q_i$ for $1 \leq i \leq n$.

This proves that $P(n + 1)$ hold in this case.

Case 2: The distance from v to w is $n + 1$. By symmetry, we can assume without loss of generality that $v = 0^{n+1}$ and $w = 1^{n+1}$.

Now by induction, there are n paths from 0^n to 1^n in H_n that don't cross in H_n. We can assume wlog[1] that each of these paths is simple.

Removing the shared first vertex, 0^n, of these paths yields paths $R_1, R_2, \ldots, R_n$. Now the R_i's are vertex disjoint except for their common endpoint, 1^n. Let s_i be the start vertex of the R_i for $1 \leq i \leq n$.

We now define $n + 1$ paths in H_{n+1} from 0^{n+1} to 1^{n+1} that don't cross.

The first of these paths will be

$$\pi_1 ::= 0^{n+1}, 10^n, 1R_1.$$

For $2 \leq i \leq n$, the ith of these paths will be

$$\pi_i ::= 0^{n+1}, 0s_i, 1R_i.$$

These paths don't cross because

- the paths $1R_i$ for $1 \leq i \leq n$ are vertex disjoint except for their common endpoint, 1^{n+1}, because the R_i's are vertex disjoint except for their common endpoint, 1^n,

[1] *without loss of generality*

- a vertex $0s_i$ does not appear on π_j for any for $j \neq i$ because the $s_i \neq s_j$ for $j \neq i$, and the other vertices on the π_j's start with 1,
- the vertex 10^n appears only on π_1. This follows because if it appeared on π_i for $i \neq 1$ it must appear on $1R_i$. That would imply that 0^n appears on R_i, contradicting the fact that the original path $0^n, R_i$ in H_n is simple.

Finally, the $n+1$st path will be

$$\pi_{n+1} ::= 0^{n+1}, 0R_1, 1^{n+1}.$$

Note that, since all but the final vertex on π_{n+1} start with 0, the only vertices besides the endpoints that π_{n+1} could share with another path would be $0s_i$ for $2 \leq i \leq n$. But none of these appear on π_{n+1} because, except for their shared endpoint, R_1 is vertex disjoint from all the other R_i's.

This proves that $P(n+1)$ holds in case 2, and therefore holds in all cases, which completes the proof by induction. ■

Note that this proof implicitly defines a recursive procedure that, for any two vertices in H_n, finds between the two vertices n simple paths of length at most $n+1$ that don't cross. ■