

In-Class Problems Week 6, Wed.

Problem 1.

For each of the following pairs of graphs, either define an isomorphism between them, or prove that there is none. (We write ab as shorthand for $a-b$.)

(a)

$$G_1 \text{ with } V_1 = \{1, 2, 3, 4, 5, 6\}, E_1 = \{12, 23, 34, 14, 15, 35, 45\}$$

$$G_2 \text{ with } V_2 = \{1, 2, 3, 4, 5, 6\}, E_2 = \{12, 23, 34, 45, 51, 24, 25\}$$

(b)

$$G_3 \text{ with } V_3 = \{1, 2, 3, 4, 5, 6\}, E_3 = \{12, 23, 34, 14, 45, 56, 26\}$$

$$G_4 \text{ with } V_4 = \{a, b, c, d, e, f\}, E_4 = \{ab, bc, cd, de, ae, ef, cf\}$$

(c)

$$G_5 \text{ with } V_5 = \{a, b, c, d, e, f, g, h\}, E_5 = \{ab, bc, cd, ad, ef, fg, gh, he, dh, bf\}$$

$$G_6 \text{ with } V_6 = \{s, t, u, v, w, x, y, z\}, E_6 = \{st, tu, uv, sv, wx, xy, yz, wz, sw, vz\}$$

Problem 2.

Definition 10.2.5. A graph is *connected* iff there is a path between every pair of its vertices.

False Claim. *If every vertex in a graph has positive degree, then the graph is connected.*

(a) Prove that this Claim is indeed false by providing a counterexample.

(b) Since the Claim is false, there must be an logical mistake in the following bogus proof. Pinpoint the *first* logical mistake (unjustified step) in the proof.

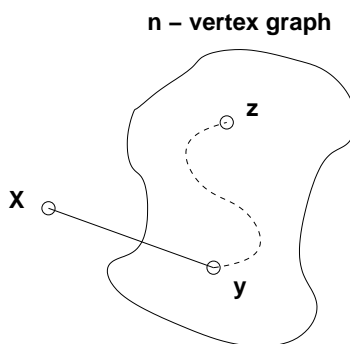
Bogus proof. We prove the Claim above by induction. Let $P(n)$ be the proposition that if every vertex in an n -vertex graph has positive degree, then the graph is connected.

Base cases: ($n \leq 2$). In a graph with 1 vertex, that vertex cannot have positive degree, so $P(1)$ holds vacuously.

$P(2)$ holds because there is only one graph with two vertices of positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

Inductive step: We must show that $P(n)$ implies $P(n+1)$ for all $n \geq 2$. Consider an n -vertex graph in which every vertex has positive degree. By the assumption $P(n)$, this graph is connected; that

is, there is a path between every pair of vertices. Now we add one more vertex x to obtain an $(n + 1)$ -vertex graph:



All that remains is to check that there is a path from x to every other vertex z . Since x has positive degree, there is an edge from x to some other vertex, y . Thus, we can obtain a path from x to z by going from x to y and then following the path from y to z . This proves $P(n + 1)$.

By the principle of induction, $P(n)$ is true for all $n \geq 0$, which proves the Claim. ■

Problem 3. (a) Prove that in every graph, there are an even number of vertices of odd degree.

Hint: The Handshaking Lemma [10.1.3](#).

(b) Conclude that at a party where some people shake hands, the number of people who shake hands an odd number of times is an even number.

(c) Call a sequence of two or more different people at the party a *handshake sequence* if, except for the last person, each person in the sequence has shaken hands with the next person in the sequence.

Suppose George was at the party and has shaken hands with an odd number of people. Explain why, starting with George, there must be a handshake sequence ending with a different person who has shaken an odd number of hands.

Hint: Just look at the people at the ends of handshake sequences that start with George.

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