

Solutions to In-Class Problems Week 6, Mon.

Problem 1.

Four Students want separate assignments to four VI-A Companies. Here are their preference rankings:

Student	Companies
Albert:	HP, Bellcore, AT&T, Draper
Rich:	AT&T, Bellcore, Draper, HP
Megumi:	HP, Draper, AT&T, Bellcore
Justin:	Draper, AT&T, Bellcore, HP

Company	Students
AT&T:	Justin, Albert, Megumi, Rich
Bellcore:	Megumi, Rich, Albert, Justin
HP:	Justin, Megumi, Albert, Rich
Draper:	Rich, Justin, Megumi, Albert

(a) Use the Mating Ritual to find *two* stable assignments of Students to Companies.

Solution. Treat Students as Boys and the result is the following assignment:

Student	Companies	Rank in the original list
Albert:	Bellcore	2
Rich:	AT&T	1
Megumi:	HP	1
Justin:	Draper	1

Treat Companies as Boys and the result is the following assignment:

Company	Students	Rank in the original list
AT&T:	Albert	2
Bellcore:	Rich	2
HP:	Megumi	2
Draper:	Justin	2



(b) Describe a simple procedure to determine whether any given stable marriage problem has a unique solution, that is, only one possible stable matching.

Solution. See if the Mating Ritual with Boys as suitors yields the same solution as the algorithm with Girls as suitors. These two marriage assignments are boy-optimal and boy-pessimal, respectively. Obviously, if every boy's optimal and pessimal choices are the same, then every boy has a unique choice. The solution is unique. ■

Problem 2.

A preserved invariant of the Mating ritual is:

For every girl, G , and every boy, B , if G is crossed off B 's list, then G has a favorite suitor and she prefers him over B .

Use the invariant to prove that the Mating Algorithm produces stable marriages. (Don't look up the proof in the Notes or slides.)

Solution. *Proof.* Let Brad be some boy and Jen be any girl that he is *not* married to on the last day of the Mating Ritual. We claim that Brad and Jen are not a rogue couple. Since Brad is an arbitrary boy, it follows that no boy is part of a rogue couple. Hence the marriages on the last day are stable.

To prove the claim, we consider two cases:

Case 1. Jen is not on Brad's list. Then by invariant P , we know that Jen prefers her husband to Brad. So she's not going to run off with Brad: the claim holds in this case.

Case 2. Otherwise, Jen is on Brad's list. But since Brad is not married to Jen, he must be choosing to serenade his wife instead of Jen, so he must prefer his wife. So he's not going to run off with Jen: the claim also holds in this case. ■

Problem 3.

Suppose that Harry is one of the boys and Alice is one of the girls in the *Mating Ritual*. Which of the properties below are preserved invariants? Why?

- a. Alice is the only girl on Harry's list.
- b. There is a girl who does not have any boys serenading her.
- c. If Alice is not on Harry's list, then Alice has a suitor that she prefers to Harry.
- d. Alice is crossed off Harry's list and Harry prefers Alice to anyone he is serenading.
- e. If Alice is on Harry's list, then she prefers to Harry to any suitor she has.

Solution. The 1st, 3rd, and 4th are preserved invariants.

- a. A preserved invariant; no girl will be added to Harry's list. If Alice got crossed off, there would be no one for Harry to marry. So she must remain as the sole girl on his list. **Reminder:** A *preserved invariant* need not be true all the time, as in this example. It only needs to stay true once it first becomes true.

- b. Not preserved; a girl may not have a suitor on the first day, —if, for example, she’s not at the top of any boy’s list —but every girl is guaranteed to have one at the end, namely, her husband.
- c. A preserved invariant; this is the basic invariant used to verify the Ritual.
- d. A preserved invariant; Harry crosses off the girls in his order of preference, so if Alice is crossed off, Harry likes her better than anybody that’s left.
- e. Not preserved. Suppose the preferences among two couples and a third boy are:

Harry: Alice, Elvira, ...
 Billy: Elvira, Alice, ...
 Wilfred: Elvira, ...
 Alice: Billy, Harry, ...
 Elvira: Wilfred, Billy, ...

The alleged invariant is true on the first day since Harry is Alice’s only suitor. But Elvira rejects Billy in favor of Wilfred on the first afternoon, so on the second day, Billy and Harry are serenading Alice. Since Alice prefers Billy to Harry, the alleged invariant is no longer true, so it was not preserved.

■

Problem 4.

Consider a stable marriage problem with 4 boys and 4 girls and the following partial information about their preferences:

B1:	G1	G2	-	-
B2:	G2	G1	-	-
B3:	-	-	G4	G3
B4:	-	-	G3	G4
G1:	B2	B1	-	-
G2:	B1	B2	-	-
G3:	-	-	B3	B4
G4:	-	-	B4	B3

(a) Verify that

$$(B1, G1), (B2, G2), (B3, G3), (B4, G4)$$

will be a stable matching whatever the unspecified preferences may be.

Solution. • $B1$ and $B2$ get their 1st choice, so won’t be in a rogue couple.

- $G1$ and $G2$ get their 2nd choices, so won’t be in a rogue couple with the other two boys, $B3$ or $B4$. So $G1$ and $G2$ won’t be in any rogue couple, either.
- $G3$ and $G4$ get their best remaining choices, so will never be in a rogue couple.

- This leaves no possible rogue partners for $B3$ and $B4$.

So the marriages are sure to be stable. ■

(b) Explain why the stable matching above is neither boy-optimal nor boy-pessimal and so will not be an outcome of the Mating Ritual.

Solution. Notice that giving $G1$ and $G2$ their first choices, that is, marrying $(B1, G2)$ and $(B2, G1)$ would also be stable for the same reason. But with this switch, $B1$ does worse. So the stable matching above is not boy-pessimal.

Likewise, after marrying off the first two boys and girls, giving $B3$ and $B4$ their best remaining choices, that is, marrying $(B3, G4)$, $(B4, G3)$, will also be stable. But with this switch, $B3$ does better. So the stable matching above is not boy-optimal.

This implies that the stable matching above would not be produced by the Mating Ritual. ■

(c) Describe how to define a set of marriage preferences among n boys and n girls which have at least $2^{n/2}$ stable assignments.

Hint: Arrange the boys into a list of $n/2$ pairs, and likewise arrange the girls into a list of $n/2$ pairs of girls. Choose preferences so that the k th pair of boys ranks the k th pair of girls just below the previous pairs of girls, and likewise for the k th pair of girls. Within the k th pairs, make sure each boy's first choice girl in the pair prefers the other boy in the pair.

Solution. Suppose a match has the two boys in the k th pair married to the two girls in the k th pair, for $1 \leq k \leq n/2$. A boy John, in the k th pair of boys will never be in a rogue couple with a girl, Jill, who is in the j th pair of girls for $j \neq k$, because if $j > k$, then Jill prefers her partner in the j th pair to John, and if $j < k$ then John prefers his partner in the k th pair to Jill.

A rogue couple can only involve a boy, John, and a girl, Mary, in the same pair, but this is impossible since (exactly) one of John and Mary must be married to their preferred choice in their pair.

Since each boy can be stably married to either of the girls in the k th pair, and there are $n/2$ pairs, the total number of such stable matchings is $2^{n/2}$. ■

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