Solutions to In-Class Problems Week 5, Fri.

Problem 1.
The Massachusetts Turnpike Authority is concerned about the integrity of the new Zakim bridge. Their consulting architect has warned that the bridge may collapse if more than 1000 cars are on it at the same time. The Authority has also been warned by their traffic consultants that the rate of accidents from cars speeding across bridges has been increasing.

Both to lighten traffic and to discourage speeding, the Authority has decided to make the bridge *one-way* and to put tolls at *both* ends of the bridge (don't laugh, this is Massachusetts). So cars will pay tolls both on entering and exiting the bridge, but the tolls will be different. In particular, a car will pay \$3 to enter onto the bridge and will pay \$2 to exit. To be sure that there are never too many cars on the bridge, the Authority will let a car onto the bridge only if the difference between the amount of money currently at the entry toll booth minus the amount at the exit toll booth is strictly less than a certain threshold amount of $\$T_0$.

The consultants have decided to model this scenario with a state machine whose states are triples of natural numbers, (A, B, C), where

- A is an amount of money at the entry booth,
- B is an amount of money at the exit booth, and
- C is a number of cars on the bridge.

Any state with $C > 1000$ is called a *collapsed* state, which the Authority dearly hopes to avoid. There will be no transition out of a collapsed state.

Since the toll booth collectors may need to start off with some amount of money in order to make change, and there may also be some number of "official" cars already on the bridge when it is opened to the public, the consultants must be ready to analyze the system started at *any* uncollapsed state. So let A_0 be the initial number of dollars at the entrance toll booth, B_0 the initial number of dollars at the exit toll booth, and $C_0 \leq 1000$ the number of official cars on the bridge when it is opened. You should assume that even official cars pay tolls on exiting or entering the bridge after the bridge is opened.

(a) Give a mathematical model of the Authority's system for letting cars on and off the bridge by specifying a transition relation between states of the form (A, B, C) above.

Solution. State (A, B, C) goes to state

(i) $(A + 3, B, C + 1)$, provided that $A - B < T_0$ and $C \leq 1000$. This transition models the case where a car enters the bridge.

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(ii) $(A, B+2, C-1)$, provided that $0 < C \leq 1000$. This transition models the case where a car leaves the bridge.

Note that the condition for the first transition has $C \leq 1000$ instead of $C < 1000$. A car can enter so long as it is not in the collapsed state ($C > 1000$). In other words, a car may still enter when $C = 1000$; and the next state will be a collapsed state with $C = 1001 > 1000$.

■

(b) Characterize each of the following derived variables

$$A, B, A+B, A-B, 3C-A, 2A-3B, B+3C, 2A-3B-6C, 2A-2B-3C$$

as one of the following

constant	C
strictly increasing	SI
strictly decreasing	SD
weakly increasing but not constant	WI
weakly decreasing but not constant	WD
none of the above	N

and briefly explain your reasoning.

Solution. In every transition, at least one of A and B increases. So their sum is strictly increasing. $2A-3B$ can fluctuate, going up on (i) and down on (ii).

The difference $3C-A$ doesn't change under transitions of type (i), but decreases under transitions of type (ii); so is weakly decreasing.

However, $B+3C$ increases under transitions of type (i), but decreases under transitions of type (ii).

On the other hand, $6C$ and $2A-3B$ simultaneously increase by 6 under transition (i) or simultaneously decrease by 6 under transition (ii), which makes their difference constant.

Finally, under (i), $2A$ increases by 6, B is unchanged, and $3C$ increases by 3, so $2A-2B-3C$ increases by $6-3=3$. However, under (ii), A is unchanged, $3C$ decreases by 3 and $2B$ increases by 4, so $2A-2B-3C$ decreases by $-(-4)-3=1$.

The completed table follows.

A	WI
B	WI
$A+B$	SI
$A-B$	N
$3C-A$	WD
$2A-3B$	N
$B+3C$	N
$2A-3B-6C$	C
$2A-2B-3C$	N

■

The Authority has asked their engineering consultants to determine T and to verify that this policy will keep the number of cars from exceeding 1000.

The consultants reason that if C_0 is the number of official cars on the bridge when it is opened, then an additional $1000 - C_0$ cars can be allowed on the bridge. So as long as $A - B$ has not increased by $3(1000 - C_0)$, there shouldn't more than 1000 cars on the bridge. So they recommend defining

$$T_0 ::= 3(1000 - C_0) + (A_0 - B_0), \tag{1}$$

where A_0 is the initial number of dollars at the entrance toll booth, B_0 is the initial number of dollars at the exit toll booth.

(c) Use the results of part (b) to define a simple predicate, P, on states of the transition system which is satisfied by the start state, that is $P(A_0, B_0, C_0)$ holds, is not satisfied by any collapsed state, and is a preserved invariant of the system. Explain why your P has these properties.

Solution. Let $D_0 ::= 2A_0 - 3B_0 - 6C_0$.

Preserved Invariant:

$$P(A, B, C) ::= [2A - 3B - 6C = D_0] \text{ AND } [C \leq 1000].$$

Note that $P(A_0, B_0, C_0)$ is true because we know that $C_0 \leq 1000$, and it is not true in any collapsed state. To verify that P is preserved, suppose state (A, B, C) has a transition to (A', B', C'), and $P(A, B, C)$ is true. We verify that $P(A', B', C')$ is true by considering the two kinds of transitions.

Transition (i) (a car enters the bridge): so

$$6C' = 6(C + 1) = 6C + 6 = (2A - 3B - D_0) + 6 = 2(A + 3) - 3B - D_0 = 2A' - 3B' - D_0,$$

which implies that

$$2A' - 3B' - 6C' = D_0, \tag{2}$$

as required.

Also, the transition is possible only if $A - B < T_0$. But this implies

$$\begin{aligned}
6C' &= 2A' - 3B' - D_0 && \text{(by (2))}\\
&= 2(A' - B') - B' - D_0 \\
&= 2((A+3) - B) - B - D_0 && \text{(since } A' = A+3,\ B' = B\text{)}\\
&= 2(A - B) - B - D_0 + 6 \\
&\leq 2(A - B) - B_0 - D_0 + 6 && \text{(since } B \text{ is WI)}\\
&\leq 2(T_0 - 1) - B_0 - D_0 + 6 && \text{(since } A - B \leq T_0 - 1\text{)}\\
&= 2[3(1000 - C_0) + (A_0 - B_0)] - B_0 - D_0 + 4 \text{(by (1))}\\
&= 6000 - 6C_0 + 2A_0 - 3B_0 - D_0 + 4 \\
&= 6004,
\end{aligned}$$

and so $C' \leq \lfloor 6004/6 \rfloor = 1000$, as required.

Transition (ii) (a car leaves the bridge): so

$$6C' = 6(C - 1) = 6C - 6 = 2A - 3B - 6 = 2A - 3(B + 2) = 2A' - 3B'.$$

In addition, $C' < C \leq 1000$ so $C' \leq 1000$. ∎

(d) A clever MIT intern working for the Turnpike Authority agrees that the Turnpike's bridge management policy will be *safe*: the bridge will not collapse. But she warns her boss that the policy will lead to *deadlock*— a situation where traffic can't move on the bridge even though the bridge has not collapsed.

Explain more precisely in terms of system transitions what the intern means, and briefly, but clearly, justify her claim.

Solution. The intern means that any long enough sequence of transitions will arrive at a state in which no transition is possible, even though there are no cars on the bridge. This happens because every time a car enters and then exits the bridge the value of $A - B$ increases by 1. So after 3000 cars have crossed the bridge, no further car can enter the bridge because

$$A - B \geq 3000 + A_0 - B_0 \geq 3(1000 - C_0) + (A_0 - B_0) = T_0.$$

After that, cars can only exit the bridge. So after at most 3000+1000 transitions, the system deadlocks with the bridge empty but no cars allowed onto the bridge.

∎

Problem 2.

In some terms when 6.042 is not taught in a TEAL room, students sit in a square arrangement during recitations. An outbreak of beaver flu sometimes infects students in recitation; beaver flu is a rare variant of bird flu that lasts forever, with symptoms including a yearning for more quizzes and the thrill of late night problem set sessions.

Here is an example of a 6×6 recitation arrangement with the locations of infected students marked with an asterisk.

*				*	
	*				
		*	*		
		*			
			*		*

Outbreaks of infection spread rapidly step by step. A student is infected after a step if either

- the student was infected at the previous step (since beaver flu lasts forever), or
- the student was adjacent to *at least two* already-infected students at the previous step.

Here *adjacent* means the students' individual squares share an edge (front, back, left or right, but *not* diagonal). Thus, each student is adjacent to 2, 3 or 4 others.

In the example, the infection spreads as shown below.

*				*	
	*				
		*	*		
		*			
			*		*

$\Rightarrow$

*	*			*	
*	*	*			
	*	*	*		
		*			
		*	*		
		*	*	*	*

$\Rightarrow$

*	*	*		*	
*	*	*	*		
*	*	*	*		
	*	*	*		
		*	*	*	
		*	*	*	*

In this example, over the next few time-steps, all the students in class become infected.

Theorem. *If fewer than n students among those in an $n \times n$ arrangment are initially infected in a flu outbreak, then there will be at least one student who never gets infected in this outbreak, even if students attend all the lectures.*

Prove this theorem.

Hint: Think of the state of an outbreak as an $n \times n$ square above, with asterisks indicating infection. The rules for the spread of infection then define the transitions of a state machine. Try to derive a weakly decreasing state variable that leads to a proof of this theorem.

Solution. *Proof.* Define the *perimeter* of an infected set of students to be the number of edges with infection on exactly one side. Let ν be size (number of edges) in the perimeter.

We claim that ν is a weakly decreasing variable. This follows because the perimeter changes after a transition only because some squares became newly infected. By the rules above, each newly-infected square is adjacent to at least two previously-infected squares. Thus, for each newly-infected square, at least two edges are removed from the perimeter of the infected region, and at most two edges are added to the perimeter. Therefore, the perimeter of the infected region cannot increase.

Now if an $n \times n$ grid is completely infected, then the perimeter of the infected region is $4n$. Thus, the whole grid can become infected only if the perimeter is initially at least $4n$. Since each square has perimeter 4, at least n squares must be infected initially for the whole grid to become infected. ■

Problem 3.
Start with 102 coins on a table, 98 showing heads and 4 showing tails. There are two ways to change the coins:

(i) flip over any ten coins, or

(ii) let n be the number of heads showing. Place $n+1$ additional coins, all showing tails, on the table.

For example, you might begin by flipping nine heads and one tail, yielding 90 heads and 12 tails, then add 91 tails, yielding 90 heads and 103 tails.

(a) Model this situation as a state machine, carefully defining the set of states, the start state, and the possible state transitions.

Solution. This can be modeled by a state machine. The state of the machine is the number of heads and tails. The start state is $(98, 4)$, and the transitions are:

$$(h,t) \to \begin{cases} (h-a+(10-a), t+a-(10-a)) & \text{for } 10 \leq h+t \& 0 \leq a \leq \min(10,h). \\ (h, t+h+1). \end{cases}$$

■

(b) Explain how to reach a state with exactly one tail showing.

Solution. One way is to:

1. Do operation 2 three times, yielding $(98, 4+3\cdot 99) = (98, 301)$.
2. Repeat 30 times: Do operation 1 to flip 10 tails into heads. This will result in the state $(398, 1)$, which is the desired state.

■

(c) Define the following derived variables:

C	$::=$	the number of coins on the table,	H	$::=$	the number of heads,
T	$::=$	the number of tails,	C_2	$::=$	remainder$(C/2)$,
H_2	$::=$	remainder$(H/2)$,	T_2	$::=$	remainder$(T/2)$.

Which of these variables is

1. strictly increasing

   Solution. NONE ■

2. weakly increasing

   Solution. C, H_2 ■

3. strictly decreasing

   Solution. NONE ■

4. weakly decreasing

   Solution. H_2 ■

5. constant

   Solution. H_2 ■

(d) Prove that it is not possible to reach a state in which there is exactly one head showing.

Solution. We claimed above that H_2 is an invariant value, that is, it does not change under state transitions. To prove this, let (h, t) be a state with h even. For the next state, we have two cases to consider:

1. The first operation is executed: $(h, t) \rightarrow (h - 2a + 10, t + 2a - 10)$. Since $-2a + 10$ is even, $H_2((h, t)) = H_2(h - 2a + 10, t + 2a - 10)$.
2. The second operation is executed: $(h, t) \rightarrow (h, t + h + 1)$. The number of heads does not change in this case, so H_2 does not change.

Since the initial number of heads, 98, is even, that is, $H_2((98, 4)) = 0$, the Invariant Method now implies that the number of heads in a reachable state is always even. But since one is odd, it is not possible to reach a state in which there is exactly one head showing.

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