Mathematics for Computer Science
MIT 6.042J/18.062J

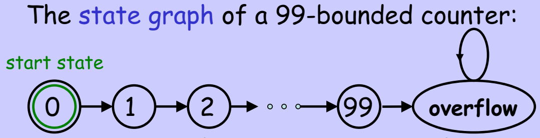
State Machines

State machines

step by step processes (may step in response to input —not today)

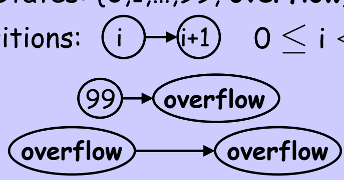
State machines

The state graph of a 99-bounded counter:



States: {0,1,...,99, **overflow**}

Transitions:



$0 \leq i < 99$

Die Hard

Image removed due to copyright restrictions.

Picture source: http://movieweb.com/movie/diehard3/

Die Hard

Simon says: On the fountain, there should be 2 jugs, do you see them? A 5-gallon and a 3-gallon. Fill one of the jugs with exactly 4 gallons of water and place it on the scale and the timer will stop. You must be precise; one ounce more or less will result in detonation. If you're still alive in 5 minutes, we'll speak.

Die Hard

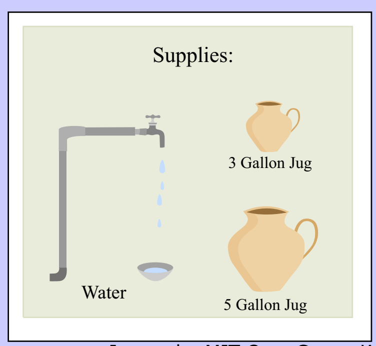
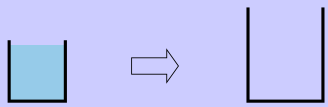


Image by MIT OpenCourseWare.

Die Hard

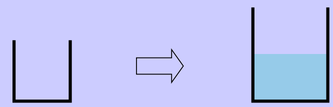
Transferring water:



3 Gallon Jug 5 Gallon Jug

Die Hard

Transferring water:



3 Gallon Jug 5 Gallon Jug

Die hard state machine

State:

amount of water in jugs: (b,l)

$0 \leq b \leq 5, 0 \leq l \leq 3$

Start State: (0,0)

State machines

Die Hard Transitions:

1. Fill little jug: (b, l) → (b, 3) for l < 3
2. Fill big jug: (b, l) → (5, l) for b < 5
3. Empty little jug: (b, l) → (b, 0) for l > 0
4. Empty big jug: (b, l) → (0, l) for b > 0

State machines

5. Pour big jug into little jug

(i) If no overflow, then (b,l)→(0,b+l)

b+l ≤ 3

(ii) otherwise (b,l) →(b−(3−l),3)

6. Pour little jug into big jug.

Likewise

Die Hard

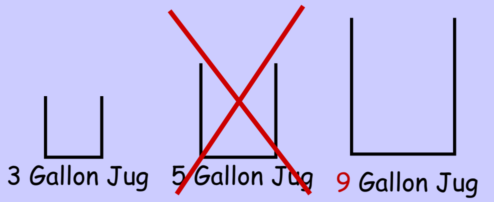
Simon's challenge:

Disarm the bomb by putting precisely 4 gallons of water on the scale, or it will blow up.

(You can figure out how)

Die Hard once and for all

What if have a 9 gallon jug instead?



Can you do it? Can you prove it?

Preserved Invariants

Die hard once and for all

preserved invariant:

P(state) ::= "3 divides the number of gallons in each jug."

$$P((b,l)) ::= (3 \mid b \text{ AND } 3 \mid l)$$

Preserved Invariants

Floyd's Invariant Method

(just like induction)

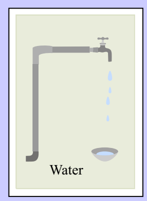
Base case: Show P(start)

Preservation case: Show

if P(q) and (q) → (r), then P(r)

Conclusion: P holds for all reachable states, including final state (if any)

Die Hard Once & For All



Corollary: No state (4,x) is reachable, so Bruce Dies!

Image by MIT OpenCourseWare.

The Diagonal Robot

the robot is on a grid

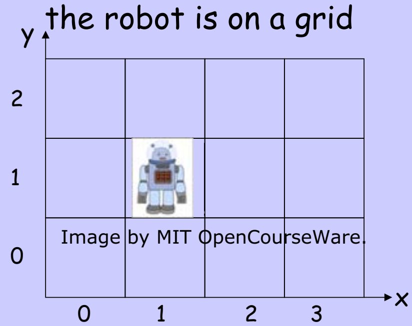
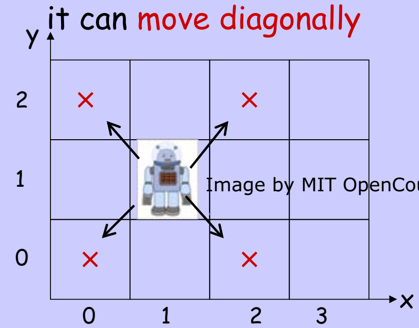


Image by MIT OpenCourseWare.

The Diagonal Robot

it can move diagonally



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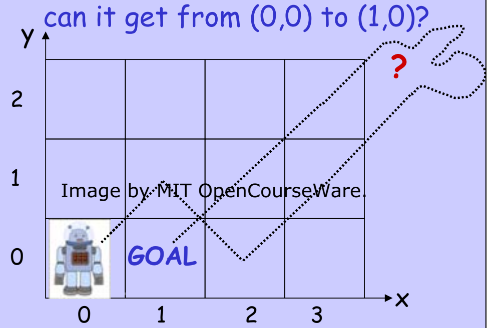
The Diagonal Robot

can it get from (0,0) to (1,0)?



Robot Preserved Invariant

NO! preserved invariant:

P((x, y)) ::= x + y is even

move adds ±1 to **both** x & y,
preserving parity of x+y.
Also, P((0, 0)) is true.

Robot Preserved Invariant

So all positions (x,y)
reachable from (0,0)
have x + y even.
But 1 + 0 = 1 is odd, so
(1,0) is not reachable.

Robert W Floyd (1934–2001)

Photograph removed due to copyright restrictions.

Eulogy by Knuth: http://www.acm.org/pubs/membernet/stories/floyd.pdf
Picture source: http://www.stanford.edu/dept/news/report/news/november7/floydobit-117.html

Team Problems

Problems
1 & 2

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Spring 2010