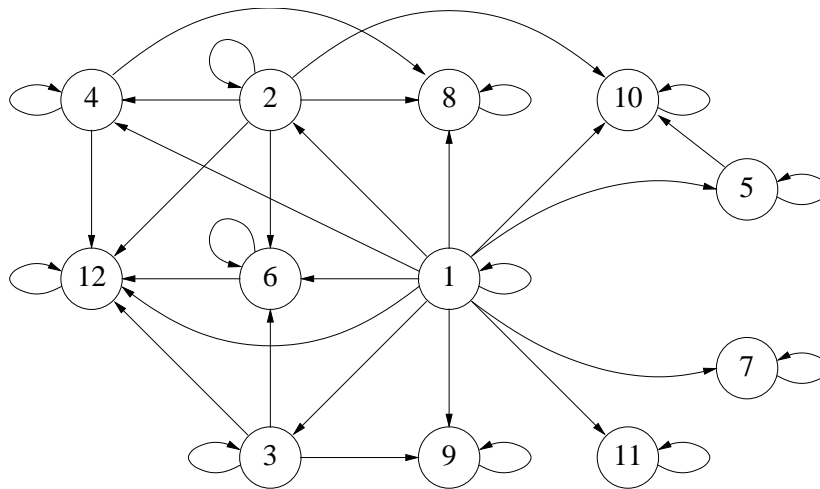


## Solutions to In-Class Problems Week 5, Mon.

### Problem 1.

If  $a$  and  $b$  are distinct nodes of a digraph, then  $a$  is said to *cover*  $b$  if there is an edge from  $a$  to  $b$  and every path from  $a$  to  $b$  traverses this edge. If  $a$  covers  $b$ , the edge from  $a$  to  $b$  is called a *covering edge*.

(a) What are the covering edges in the following DAG?



**Solution.** TBA ■

(b) Let  $\text{covering}(D)$  be the subgraph of  $D$  consisting of only the covering edges. Suppose  $D$  is a finite DAG. Explain why  $\text{covering}(D)$  has the same positive path relation as  $D$ .

*Hint:* Consider *longest* paths between a pair of vertices.

**Solution.** What we need to show is that if there is a path in  $D$  between vertices  $a \neq b$ , then there is a path consisting only of covering edges from  $a$  to  $b$ . But since  $D$  is a finite DAG, there must be a *longest* path from  $a$  to  $b$ . Now every edge on this path must be a covering edge or it could be replaced by a path of length 2 or more, yielding a longer path from  $a$  to  $b$ . ■

(c) Show that if two DAG's have the same positive path relation, then they have the same set of covering edges.

**Solution.** *Proof.* Suppose  $C$  and  $D$  are DAG's with the same positive path relation and that  $a \rightarrow b$  is a covering edge of  $C$ . We want to show that  $a \rightarrow b$  must also be a covering edge of  $D$ .

Since  $a \rightarrow b$  itself defines a (length one) positive length path in  $C$ , there must be a positive length path in  $D$  from  $a$  to  $b$ . If this positive length path in  $D$  is of length greater than one, then the path must consist of a positive length path from  $a$  to  $c$  followed by a positive length path from  $c$  to  $b$  for some vertex,  $c$ . Also, since  $D$  is a DAG,  $c$  cannot be  $a$  or  $b$ .

This means there must also be positive length paths in  $C$  from  $a$  to  $c$  and from  $c$  to  $b$ , and neither of these paths can traverse  $a \rightarrow b$  or there would be a cycle. Hence the path from  $a$  to  $c$  to  $b$  is a path in  $C$  that does not traverse  $a \rightarrow b$ , contradicting the fact that  $a \rightarrow b$  is a covering edge of  $C$ .

In sum, there is a length one path from  $a$  to  $b$  in  $D$ , namely  $a \rightarrow b$ , and this is the *only* path from  $a$  to  $b$  in  $D$ , which proves that  $a \rightarrow b$  is a covering edge in  $D$ . ■

(d) Conclude that covering( $D$ ) is the *unique* DAG with the smallest number of edges among all digraphs with the same positive path relation as  $D$ .

**Solution.** By part (c), any DAG with the same positive path relation as  $D$  must contain all the edges of covering( $D$ ). By part (b), covering( $D$ ) has this same positive path relation. It follows immediately that covering( $D$ ) is the unique minimum-size DAG with the same positive path relation as  $D$ . ■

The following examples show that the above results don't work in general for digraphs with cycles.

(e) Describe two graphs with vertices  $\{1, 2\}$  which have the same set of covering edges, but not the same positive path relation (*Hint*: Self-loops.)

**Solution.** Let one graph have edges  $\{(1, 2), (1, 1)\}$  and the other  $\{(1, 2), (2, 2)\}$ . They have the same set of covering edges, namely,  $(1, 2)$ . But in the second there is a positive length path from 2 to 2, namely a path of length one but there is no positive length path from 2 to 2 in the first graph. ■

- (f) (i) The *complete digraph* without self-loops on vertices 1, 2, 3 has edges between every two distinct vertices. What are its covering edges?  
 (ii) What are the covering edges of the graph with vertices 1, 2, 3 and edges  $1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1$ ?  
 (iii) What about their positive path relations?

**Solution.** (i) There are no covering edges, since there is always a length two path from  $a$  to  $b$  that does not use the edge  $a \rightarrow b$ .

- (ii) All three edges are the covering edges.  
 (iii) They have the same positive path relation, namely, each vertex is connected to all the vertices, including itself, by positive length paths. ■

**Problem 2. (a)** Give an example showing that two vertices in a digraph may be on the same cycle, but *not* necessarily on the same *simple* cycle.

**Solution.** Let the vertices be  $a, b, c$  and edges be  $(a, b), (b, a), (b, c), (c, b)$ . Now  $a$  and  $c$  are on the cycle  $a, b, c, b, a$ , but every cycle from  $a$  to  $c$  must go through  $b$  at least twice, and so will not be simple. ■

**(b)** Prove that if two vertices in a digraph are connected, then they are connected by a simple path. *Hint:* the shortest path.

**Solution.** Consider a shortest path from  $a$  to  $b \neq a$ :

$$a = a_0, a_1, \dots, a_i, \dots, a_j, \dots, a_k = b,$$

and suppose this path is not simple. That is, suppose  $a_i = a_j$  for some  $i, j$ . Then

$$a = a_0, a_1, \dots, a_i, a_{j+1}, \dots, a_k = b.$$

is a shorter path from  $a$  to  $b$ , a contradiction. ■

### Problem 3.

In an  $n$ -player *round-robin tournament*, every pair of distinct players compete in a single game. Assume that every game has a winner—there are no ties. The results of such a tournament can then be represented with a *tournament digraph* where the vertices correspond to players and there is an edge  $x \rightarrow y$  iff  $x$  beat  $y$  in their game.

**(a)** Explain why a tournament digraph cannot have cycles of length 1 or 2.

**Solution.** There are no self-loops in a tournament graph since no player plays himself, so no length 1 cycles. Also, it cannot be that  $x$  beats  $y$  and  $y$  beats  $x$  for  $x \neq y$ , since every pair competes exactly once and there are no ties. This means there are no length 2 cycles. ■

**(b)** Is the “beats” relation for a tournament graph always/sometimes/never:

- asymmetric?
- reflexive?
- irreflexive?
- transitive?

Explain.

**Solution.** No self-loops implies the relation is irreflexive. It is also asymmetric since it is irreflexive and for every pair of distinct players, exactly one game is played and results in a win for one of the players. Some tournament graphs represent transitive relations and others don’t. ■

(c) Show that a tournament graph represents a total order iff there are no cycles of length 3.

**Solution.** As observed in the previous part, the “beats” relation whose graph is a tournament is asymmetric and irreflexive. Since every pair of players is comparable, the relation is a total order iff it is transitive.

“Beats” is transitive iff for any players  $x, y$  and  $z$ ,  $x \rightarrow y$  and  $y \rightarrow z$  implies that  $x \rightarrow z$  (and consequently that there is no edge  $z \rightarrow x$ ). Therefore, “beats” is transitive iff there are no cycles of length 3. ■

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