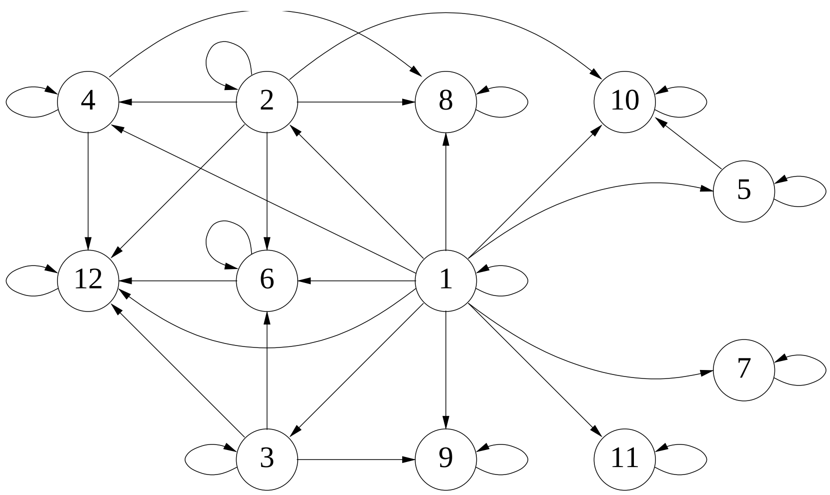
In-Class Problems Week 5, Mon.

Problem 1.
If a and b are distinct nodes of a digraph, then a is said to *cover* b if there is an edge from a to b and every path from a to b traverses this edge. If a covers b, the edge from a to b is called a *covering edge*.

(a) What are the covering edges in the following DAG?



(b) Let covering (D) be the subgraph of D consisting of only the covering edges. Suppose D is a finite DAG. Explain why covering (D) has the same positive path relation as D.

Hint: Consider *longest* paths between a pair of vertices.

(c) Show that if two *DAG*'s have the same positive path relation, then they have the same set of covering edges.

(d) Conclude that covering (D) is the *unique* DAG with the smallest number of edges among all digraphs with the same positive path relation as D.

The following examples show that the above results don't work in general for digraphs with cycles.

(e) Describe two graphs with vertices $\{1, 2\}$ which have the same set of covering edges, but not the same positive path relation (*Hint:* Self-loops.)

(f) (i) The *complete digraph* without self-loops on vertices $1, 2, 3$ has edges between every two distinct vertices. What are its covering edges?

(ii) What are the covering edges of the graph with vertices $1, 2, 3$ and edges $1 \to 2, 2 \to 3, 3 \to 1$?

(iii) What about their positive path relations?

Problem 2. (a) Give an example showing that two vertices in a digraph may be on the same cycle, but *not* necessarily on the same *simple* cycle.

(b) Prove that if two vertices in a digraph are connected, then they are connected by a simple path. *Hint:* the shortest path.

Problem 3.

In an n-player *round-robin tournament*, every pair of distinct players compete in a single game. Assume that every game has a winner —there are no ties. The results of such a tournament can then be represented with a *tournament digraph* where the vertices correspond to players and there is an edge $x \to y$ iff x beat y in their game.

(a) Explain why a tournament digraph cannot have cycles of length 1 or 2.

(b) Is the "beats" relation for a tournament graph always/sometimes/never:

- asymmetric?
- reflexive?
- irreflexive?
- transitive?

Explain.

(c) Show that a tournament graph represents a total order iff there are no cycles of length 3.

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