Mathematics for Computer Science
MIT 6.042J/18.062J

Directed Graphs

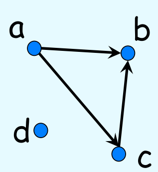
Digraphs

- a set, V, of vertices
- a set, $E \subseteq V \times V$ of directed edges

$(v,w) \in E$ notation: $v \rightarrow w$

v •⟶• w

Relations and Graphs

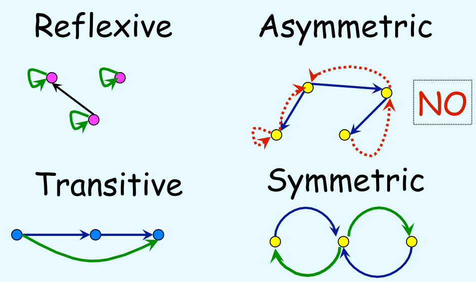


V= {a,b,c,d}
E = {(a,b), (a,c), (c,b)}

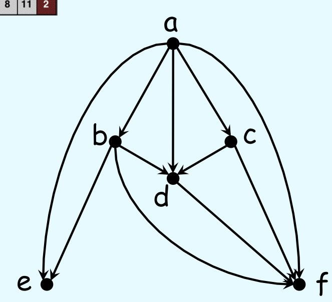
Digraphs

Formally, a digraph with vertices V is *the same* as a binary relation on V.

Graphical Properties of Relations



Graph of Strict Partial Order



Graph of Strict Partial Order

how to check?

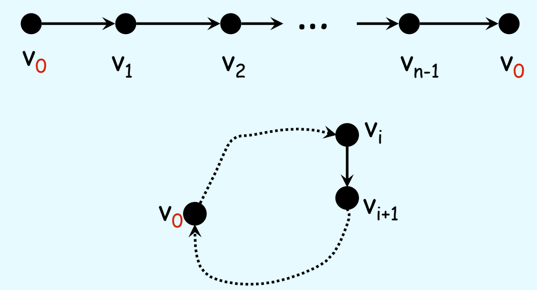
- no self-loops $i \rightarrow i \notin E$
  (irreflexive)
- if edges $i \rightarrow j$ and $j \rightarrow k$
  then shortcut edge $i \rightarrow k$ is there too
  (transitive)

Cycles

A cycle is a positive length directed path that starts and ends at the same vertex.

simple cycle: each vertex only once, except start = end

Directed Cycle



Graph of Strict Partial Order

by asymmetry: if there is a path from a to b, then there is none from b to a

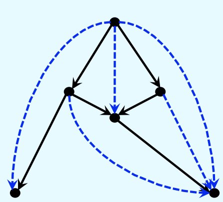
graph has no cycle

a directed acyclic graph

DAG

Graph of Strict Partial Order

strict p.o. implies DAG, but not every DAG is strict p.o.



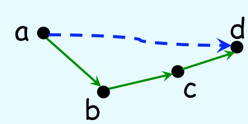
not transitive

also need these edges

Strict P.O. from a DAG

but from any DAG, get a strict p.o. by adding "transitive" edges:

if there is a path in the DAG, add an edge from start to end:



Positive Path Relation

relation R on a set V

aR⁺b iff
there is a nonzero directed path from a to b

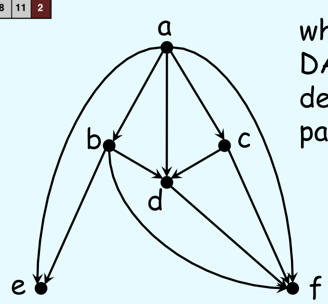
$$aRv_1Rv_2R\cdots Rb$$

DAG's & Partial Orders

Theorem:

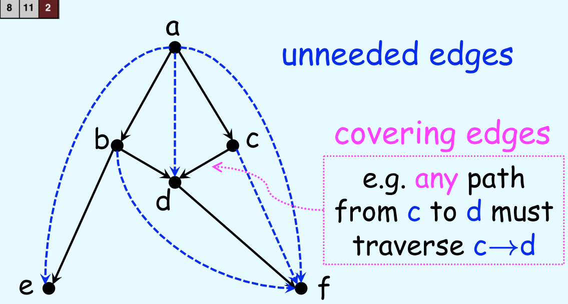
- The graph of a strict partial order is a DAG.
- The positive path relation of a DAG is a strict partial order.

Graph of Strict Partial Order



what is *smallest* DAG whose paths define this partial order?

Covering Edges



Problems

1 – 3