

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

# Directed Graphs



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# Digraphs

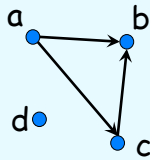
- a set,  $V$ , of vertices
- a set,  $E \subseteq V \times V$  of directed edges

$(v,w) \in E$  notation:  $v \rightarrow w$



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# Relations and Graphs



$V = \{a, b, c, d\}$   
 $E = \{(a, b), (a, c), (c, b)\}$



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# Digraphs

Formally, a digraph with vertices  $V$  is the same as a binary relation on  $V$ .



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## Graphical Properties of Relations

Reflexive

Asymmetric

Transitive

Symmetric

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## Graph of Strict Partial Order

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## Graph of Strict Partial Order

how to check?

- no self-loops  $i \rightarrow i \notin E$   
(irreflexive)
- if edges  $i \rightarrow j$  and  $j \rightarrow k$   
then shortcut edge  $i \rightarrow k$  is  
there too  
(transitive)

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## Cycles

A **cycle** is a positive length **directed** path that starts and ends at the same vertex.

**simple cycle**: each vertex only once, except start = end

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## Directed Cycle

$v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_{n-1} \rightarrow v_0$

$v_0 \rightarrow v_i \rightarrow v_{i+1} \rightarrow v_0$

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## Graph of Strict Partial Order

by **asymmetry**: if there is a path from **a** to **b**, then there is **none** from **b** to **a**

graph has no cycle,  
a directed **acyclic graph**

# DAG

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## Graph of Strict Partial Order

strict p.o. implies DAG, **but** not every DAG is strict p.o.

not transitive

also need these edges

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## Strict P.O. from a DAG

but from any DAG, get a strict p.o. by adding "transitive" edges:

if there is a **path** in the DAG, add an **edge** from start to end:

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### Positive Path Relation

relation  $R$  on a set  $V$

$aR^+b$  iff  
there is a nonzero directed  
path from  $a$  to  $b$

$$aRv_1Rv_2R\cdots Rb$$



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### DAG's & Partial Orders

Theorem:

- The graph of a strict partial order is a DAG.
- The positive path relation of a DAG is a strict partial order.

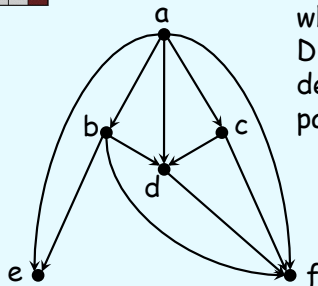


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### Graph of Strict Partial Order



what is *smallest*  
DAG whose paths  
define this  
partial order?

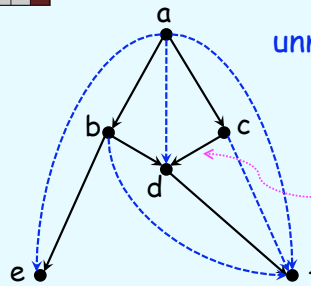


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### Covering Edges



unused edges

covering edges

e.g. any path  
from  $c$  to  $d$  must  
traverse  $c \rightarrow d$



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# Problems

## 1 - 3



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