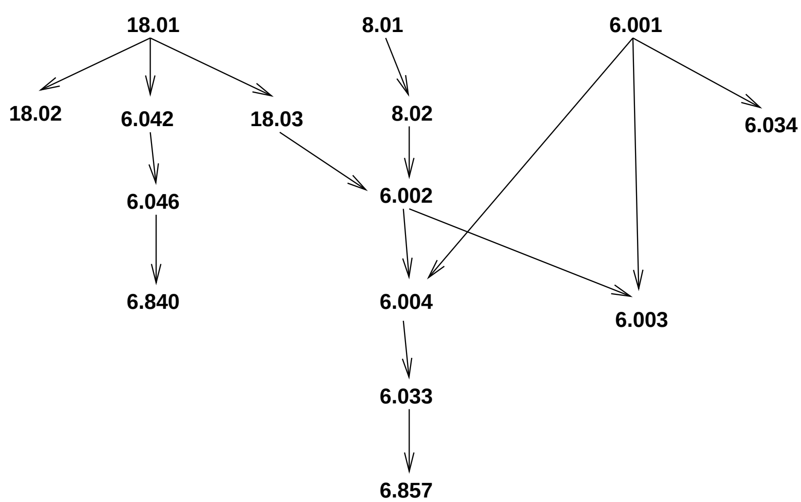
Solutions to In-Class Problems Week 4, Fri.

Problem 1.
The table below lists some prerequisite information for some subjects in the MIT Computer Science program (in 2006). This defines an indirect prerequisite relation, $\prec$, that is a strict partial order on these subjects.

$$18.01 \to 6.042 \qquad 18.01 \to 18.02$$
$$18.01 \to 18.03 \qquad 6.046 \to 6.840$$
$$8.01 \to 8.02 \qquad 6.001 \to 6.034$$
$$6.042 \to 6.046 \qquad 18.03, 8.02 \to 6.002$$
$$6.001, 6.002 \to 6.003 \qquad 6.001, 6.002 \to 6.004$$
$$6.004 \to 6.033 \qquad 6.033 \to 6.857$$

(a) Explain why exactly six terms are required to finish all these subjects, if you can take as many subjects as you want per term. Using a *greedy* subject selection strategy, you should take as many subjects as possible each term. Exhibit your complete class schedule each term using a greedy strategy.

Solution. It helps to have a diagram of the direct prerequisite relation:



There is a $\prec$-chain of length six:

$$8.01 \prec 8.02 \prec 6.002 \prec 6.004 \prec 6.033 \prec 6.857$$

So six terms are necessary, because at most one of these subjects can be taken each term.

There is no longer chain, so with the greedy strategy you will take six terms. Here are the subjects you take in successive terms.

1:	6.001	8.01	18.01		
2:	6.034	6.042	8.02	18.02	18.03
3:	6.002	6.046			
4:	6.003	6.004	6.840		
5:	6.033				
6:	6.857				

■

(b) In the second term of the greedy schedule, you took five subjects including 18.03. Identify a set of five subjects not including 18.03 such that it would be possible to take them in any one term (using some nongreedy schedule). Can you figure out how many such sets there are?

Solution. We're looking for an antichain in the $\prec$ relation that does not include 18.03. Every such antichain will have to include 18.02, 6.003, 6.034. Then a fourth subject could be any of 6.042, 6.046, and 6.840. The fifth subject could then be any of 6.004, 6.033, and 6.857. This gives a total of nine antichains of five subjects. ■

(c) Exhibit a schedule for taking all the courses —but only one per term.

Solution. We're asking for a topological sort of $\prec$. There are many. One is 18.01, 8.01, 6.001, 18.02, 6.042, 18.03, 8.02, 6.034, 6.046, 6.002, 6.840, 6.004, 6.003, 6.033, 6.857. ■

(d) Suppose that you want to take all of the subjects, but can handle only two per term. Exactly how many terms are required to graduate? Explain why.

Solution. There are $\lceil 15/2 \rceil = 8$ terms necessary. The schedule below shows that 8 terms are sufficient as well:

1:	18.01	8.01
2:	6.001	18.02
3:	6.042	18.03
4:	8.02	6.034
5:	6.046	6.002
6:	6.840	6.004
7:	6.003	6.033
8:	6.857	

■

(e) What if you could take three subjects per term?

Solution. From part (a) we know six terms are required even if there is no limit on the number of subjects per term. Six terms are also sufficient, as the following schedule shows:

1:	18.01	8.01	6.001
2:	6.042	18.03	8.02
3:	18.02	6.046	6.002
4:	6.004	6.003	6.034
5:	6.840	6.033	
6:	6.857		

■

Problem 2.
A pair of 6.042 TAs, Liz and Oscar, have decided to devote some of their spare time this term to establishing dominion over the entire galaxy. Recognizing this as an ambitious project, they worked out the following table of tasks on the back of Oscar's copy of the lecture notes.

1. **Devise a logo** and cool imperial theme music - 8 days.
2. **Build a fleet** of Hyperwarp Stardestroyers out of eating paraphernalia swiped from Lobdell - 18 days.
3. **Seize control** of the United Nations - 9 days, after task #1.
4. **Get shots** for Liz's cat, Tailspin - 11 days, after task #1.
5. **Open a Starbucks chain** for the army to get their caffeine - 10 days, after task #3.
6. **Train an army** of elite interstellar warriors by dragging people to see *The Phantom Menace* dozens of times - 4 days, after tasks #3, #4, and #5.
7. **Launch the fleet** of Stardestroyers, crush all sentient alien species, and establish a Galactic Empire - 6 days, after tasks #2 and #6.
8. **Defeat Microsoft** - 8 days, after tasks #2 and #6.

We picture this information in Figure 1 below by drawing a point for each task, and labelling it with the name and weight of the task. An edge between two points indicates that the task for the higher point must be completed before beginning the task for the lower one.

(a) Give some valid order in which the tasks might be completed.

Solution. We can easily find several of them. The most natural one is valid, too: #1, #2, #3, #4, #5, #6, #7, #8.

■

Liz and Oscar want to complete all these tasks in the shortest possible time. However, they have agreed on some constraining work rules.

- Only one person can be assigned to a particular task; they can not work together on a single task.

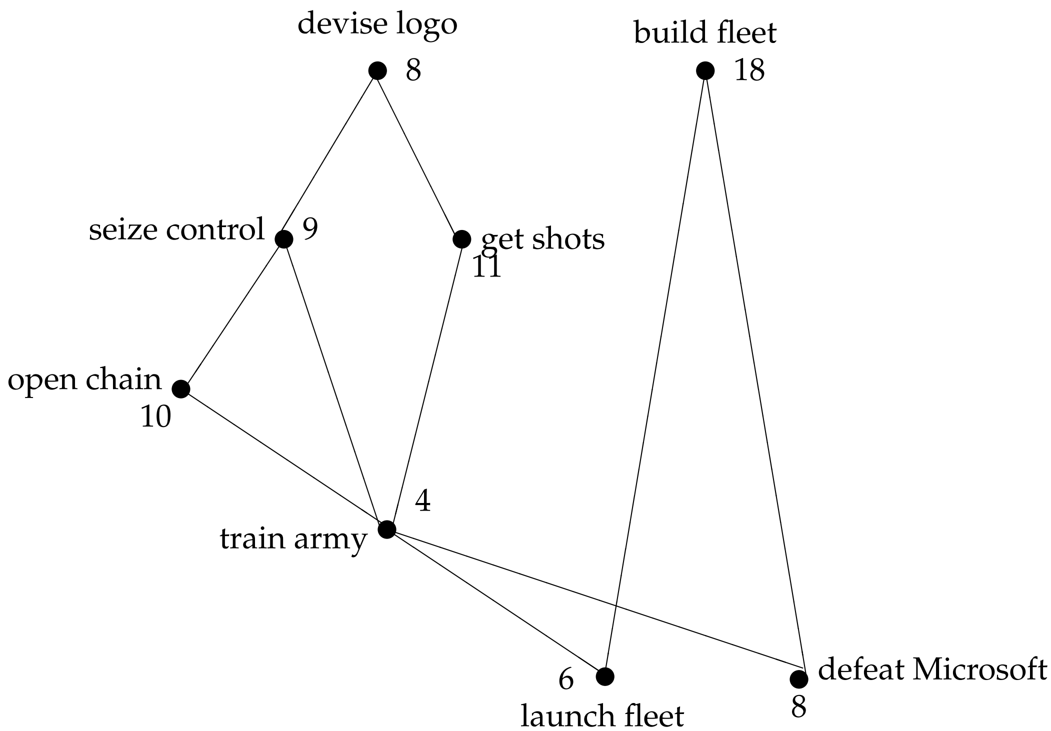


Figure 1: Graph representing the task precedence constraints.

- Once a person is assigned to a task, that person must work exclusively on the assignment until it is completed. So, for example, Liz cannot work on building a fleet for a few days, run to get shots for Tailspin, and then return to building the fleet.

(b) Liz and Oscar want to know how long conquering the galaxy will take. Oscar suggests dividing the total number of days of work by the number of workers, which is two. What lower bound on the time to conquer the galaxy does this give, and why might the actual time required be greater?

Solution.

$$\frac{8+18+9+11+10+4+6+8}{2} = 37 \text{ days}$$

If working together and interrupting work on a task were permitted, then this answer would be correct. However, the rules may prevent Liz and Oscar from both working all the time. For example, suppose the only task was building the fleet. It will take 18 days, not 18/2 days, to complete, because only one person can work on it and the other must sit idle. ■

(c) Liz proposes a different method for determining the duration of their project. He suggests looking at the duration of the "critical path", the most time-consuming sequence of tasks such that each depends on the one before. What lower bound does this give, and why might it also be too low?

Solution. The longest sequence of tasks is devising a logo (8 days), seizing the U.N. (9 days), opening a Starbucks (10 days), training the army (4 days), and then defeating Microsoft (8 days). Since these tasks must be done sequentially, galactic conquest will require at least 39 days.

If there were enough workers, this answer would be correct; however, with only two workers, Liz and Oscar may be unable to make progress on the critical path every day. For example, suppose there were only four tasks: devise logo, build fleet, seize control, get shots. Now the critical path consists of two critical tasks: devise logo, get shots, which take 19 days. But to get through this path in 19 days, some worker must be working on a critical task at all times for the 19 days. This leaves only one worker free to complete building the fleet and seizing control, which will take at least 27 days. So in fact, 27 days is the minimum time for two workers to complete these four tasks. ■

(d) What is the minimum number of days that Liz and Oscar need to conquer the galaxy? No proof is required.

Solution. 40 days. Tasks could be divided as follows:

Oscar: #1 (days 1-8), #3 (days 9-17), #4 (days 18-28), #8 (days 33-40).

Liz: #2 (days 1-18), #5 (days 19-28), #6 (days 29-32), #7 (days 33-38).

It takes some care to verify that 40 days is the best you can do. If someone comes up with a simple proof of this, tell the course staff.

■

Problem 3. **(a)** What are the maxim*al* and minim*al* elements, if any, of the power set $\mathcal{P}(\{1, \ldots, n\})$, where n is a positive integer, under the *empty relation*?

Solution. The power set is a red herring. With an empty relation on any set, every element is maximal and minimal. ■

(b) What are the maxim*al* and minim*al* elements, if any, of the set, $\mathbb{N}$, of all nonnegative integers under divisibility? Is there a minim*um* or maxim*um* element?

Solution. The minimum (and therefore unique minimal) element is 1 since 1 divides all natural numbers. The maximum (and therefore unique maximal) element is 0 since all numbers divide 0. ■

(c) What are the minimal and maximal elements, if any, of the set of integers greater than 1 under divisibility?

Solution. All prime numbers are minimal elements, since no numbers divide them.

There is no maximal element, because for any $n > 1$, there is a "larger" number under the divisibility partial order, for example, $2n$. ■

(d) Describe a partially ordered set that has no minimal or maximal elements.

Solution. $\mathbb{Z}$, $\mathbb{R}$, etc. ■

(e) Describe a partially ordered set that has a *unique minimal* element, but no minimum element. *Hint:* It will have to be infinite.

Solution. $\mathbb{Z} \cup \{i\}$ where i is a root of -1, under the usual order $\mathbb{Z}$. So i is incomparable to everything but itself, and is therefore minimal —and maximal too. The remaining elements are the integers, and none of them are minimal since $n - 1 < n$, which makes i unique. ■