

Solutions to In-Class Problems Week 4, Wed.

Problem 1.

Direct Prerequisites	Subject
18.01	6.042
18.01	18.02
18.01	18.03
8.01	8.02
8.01	6.01
6.042	6.046
18.02, 18.03, 8.02, 6.01	6.02
6.01, 6.042	6.006
6.01	6.034
6.02	6.004

(a) For the above table of MIT subject prerequisites, draw a diagram showing the subject numbers with a line going down to every subject from each of its (direct) prerequisites.

Solution.

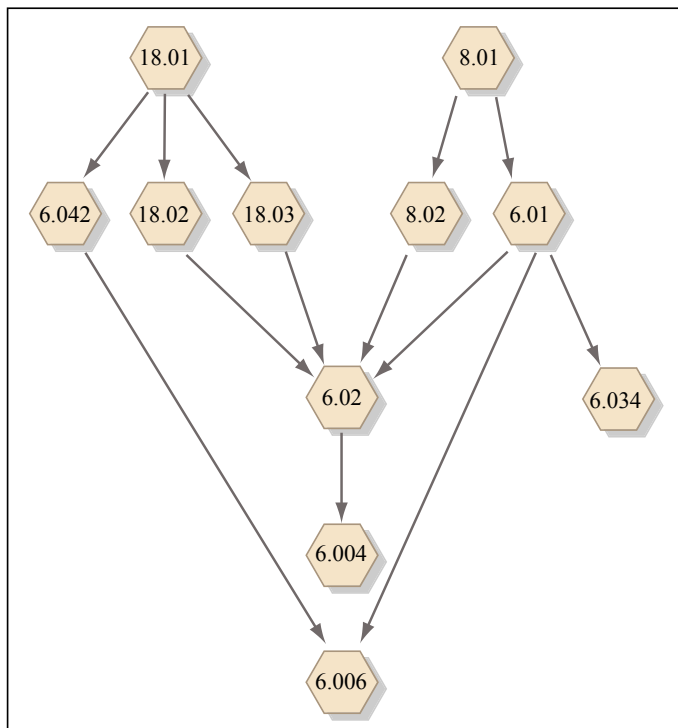


Image by MIT OpenCourseWare. ■

(b) Give an example of a collection of sets partially ordered by the proper subset relation, \subset , that is isomorphic to (“same shape as”) the prerequisite relation among MIT subjects from part (a).

Solution. For each subject, S , let

$$\text{preset}(S) ::= \{S' \mid S' \text{ is an indirect prerequisite of } S \text{ OR } S' = S\}.$$

For example,

subject	preset
18.02	{18.01, 18.02}
18.03	{18.01, 18.03}
6.006	{6.042, 18.01, 6.01, 8.01, 6.006}

Note that the “OR $S' = S$ ” clause is necessary: if we let the set representing subject S just be the indirect prerequisites of S , then 18.02 and 18.03, for example, would be represented by the same set, {18.01}. Then the correspondence between subjects and sets would no longer be a bijection, which is a requirement for isomorphism. ■

(c) Explain why the empty relation is a strict partial order and describe a collection of sets partially ordered by the proper subset relation that is isomorphic to the empty relation on five elements—that is, the relation under which none of the five elements is related to anything.

Solution. An empty relation is always a partial order: it is *vacuously* asymmetric and transitive. It's not weak because it is not reflexive; in fact it's irreflexive.

Letting the five elements be 1, 2, 3, 4, 5, the recipe of mapping an element to its preimages under the relation, with the element itself thrown in, gives the five sets $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$.

Of course any 5 sets none of which is contained in any of the others will also work, for example, all the size 4 subsets of $\{1, 2, 3, 4, 5\}$ ■

(d) Describe a *simple* collection of sets partially ordered by the proper subset relation that is isomorphic to the "properly contains" relation, \supset , on $\mathcal{P}\{1, 2, 3, 4\}$.

Solution. The standard inverse image solution involves sets of subsets. A more elegant correspondence is to let each set $A \subseteq \{1, 2, 3, 4\}$ correspond to its complement. That is,

$$f(A) = \bar{A} ::= \{1, 2, 3, 4\} - A.$$

This works because $A \supset B$ iff $\bar{A} \subset \bar{B}$ ■

Problem 2.

A binary relation, R , on a set, A , is *irreflexive* iff NOT($a R a$) for all $a \in A$. Prove that if a binary relation on a set is transitive and irreflexive, then it is strict partial order.

Solution. *Proof.* Suppose R transitive and irreflexive. Since it is transitive, to check that it is a strict partial order, we need only verify that it is asymmetric.

To prove that it is asymmetric, suppose $a R b$ holds for some $a, b \in A$. We need to prove NOT($b R a$).

So assume to the contrary that $b R a$ holds. Then $a R b$ and $b R a$, so by transitivity, $a R a$, contradicting the fact that R is irreflexive. So $b R a$ does not hold, as claimed. ■

Problem 3.

How many binary relations are there on the set $\{0, 1\}$?

How many are there that are transitive?, ... asymmetric?, ... reflexive?, ... irreflexive?, ... strict partial orders?, ... weak partial orders?

Hint: There are easier ways to find these numbers than listing all the relations and checking which properties each one has.

Solution. There are $2^4 = 16$ such relations, since in any such relation there are four possible arrows between $\{0, 1\}$ and itself, each of which may or may not be there.

There are 3 intransitive transitive relations, because the only way transitivity can fail in a relation on two elements is when there is an arrow in both directions between the elements, but one or the other or both the elements are missing a *self-loop*, that is, an arrow that starts and ends at the element. So there are $13 = 16 - 3$ transitive relations.

There are 3 asymmetric relations. Asymmetry implies no self-loops, and at most one of the two possible arrows between 0 and 1. So the only 3 possibilities are no arrows, arrow from 0 to 1, arrow from 1 to 0.

There are 4 reflexive relations, because two of the four possible arrows (the self-loops) must be present, the remaining two arrows can be either present or not present, which yields 2^2 relations. There are 4 irreflexive relations for the same reason.

There are 3 strict partial orders, because the 3 asymmetric relations are all transitive.

There are 3 weak partial orders, because the 3 strict partial orders remain distinct after adding self-loops to both elements.



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