

## Solutions to In-Class Problems Week 4, Mon.

### Problem 1.

Prove by induction:

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}, \quad (1)$$

for all  $n > 1$ .

**Solution.** *Proof.* (By Induction). The induction hypothesis is  $P(n)$  is the inequality (1).

**Base Case:** ( $n = 2$ ). The LHS of (1) in this case is  $1 + 1/4$  and the RHS is  $2 - 1/2$ . Since LHS =  $5/4 < 6/4 = 3/2 =$  RHS, inequality (1) holds, and  $P(2)$  is proved.

**Inductive Step:** Let  $n$  be any natural number greater than 1, and assume  $P(n)$  in order to prove  $P(n + 1)$ . That is, we assume (1) Adding  $1/(n + 1)^2$  to both sides of this inequality yields

$$\begin{aligned} 1 + \frac{1}{4} + \cdots + \frac{1}{n^2} + \frac{1}{(n + 1)^2} &< 2 - \frac{1}{n} + \frac{1}{(n + 1)^2} \\ &= 2 - \left( \frac{1}{n} - \frac{1}{(n + 1)^2} \right) \\ &= 2 - \left( \frac{n^2 + 2n + 1 - n}{n(n + 1)^2} \right) \\ &= 2 - \frac{n^2 + n}{n(n + 1)^2} - \frac{1}{n(n + 1)^2} \\ &= 2 - \frac{1}{n + 1} - \frac{1}{n(n + 1)^2} \\ &< 2 - \frac{1}{n + 1}. \end{aligned}$$

So we have proved  $P(n + 1)$ . ■

**Problem 2. (a)** Prove by induction that a  $2^n \times 2^n$  courtyard with a  $1 \times 1$  statue of Bill in a corner can be covered with L-shaped tiles. (Do not assume or reprove the (stronger) result of Theorem 6.1.2 that Bill can be placed anywhere. The point of this problem is to show a different induction hypothesis that works.)

**Solution.** Let  $P(n)$  be the proposition Bill can be placed in a corner of a  $2^n \times 2^n$  courtyard with a proper tiling of the remainder with L-shaped tiles.

**Base case:**  $P(0)$  is true because Bill fills the whole courtyard.

**Inductive step:** Assume that  $P(n)$  is true for some  $n \geq 0$ ; that is, there exists a tiling of the  $2^n \times 2^n$  courtyard leaving Bill in a corner.

To prove,  $P(n + 1)$ , Divide the  $2^{n+1} \times 2^{n+1}$  courtyard into four quadrants, each  $2^n \times 2^n$ . One quadrant will contain the corner designated for Bill. By induction hypothesis, we can get Bill into some corner of the quadrant, which means we can actually get him into *any* desired corner of the quadrant by rotating the tiling of the quadrant. So place Bill in the designated corner of the quadrant, and tile the rest of the quadrant.

Now tile the remaining three quadrants, leaving a tile space open in the quadrant corners that are in the middle of the whole  $2^{n+1} \times 2^{n+1}$  courtyard (as in the diagram in the proof of Theorem 6.1.2). These three spaces form an L-shape that that can be filled with a single L-shaped tile, completing the full courtyard tiling. This proves  $P(n + 1)$ , completing the proof by induction that a square courtyard with side length any power of 2 can be tiled with Bill in a corner. ■

(b) Use the result of part (a) to prove the original claim that there is a tiling with Bill in the middle.

**Solution.** To put Bill in the middle, tile each of the four quadrants, leaving the empty corner of the quadrant in the middle of the full courtyard. This leaves the four central squares of the full courtyard empty, so fill three of these squares with an L-shaped tile. This leaves a single central square untiled for Bill. ■

### Problem 3.

Find the flaw in the following bogus proof that  $a^n = 1$  for all nonnegative integers  $n$ , whenever  $a$  is a nonzero real number.

*Bogus proof.* The proof is by induction on  $n$ , with hypothesis

$$P(n) ::= \forall k \leq n. a^k = 1,$$

where  $k$  is a nonnegative integer valued variable.

**Base Case:**  $P(0)$  is equivalent to  $a^0 = 1$ , which is true by definition of  $a^0$ . (By convention, this holds even if  $a = 0$ .)

**Inductive Step:** By induction hypothesis,  $a^k = 1$  for all  $k \in \mathbb{N}$  such that  $k \leq n$ . But then

$$a^{n+1} = \frac{a^n \cdot a^n}{a^{n-1}} = \frac{1 \cdot 1}{1} = 1,$$

which implies that  $P(n + 1)$  holds. It follows by induction that  $P(n)$  holds for all  $n \in \mathbb{N}$ , and in particular,  $a^n = 1$  holds for all  $n \in \mathbb{N}$ . ■

**Solution.** The flaw comes in the inductive step, where we implicitly assume  $n \geq 1$  in order to talk about  $a^{n-1}$  in the denominator (otherwise the exponent is not a nonnegative integer, so we cannot apply the inductive hypothesis). We checked the base case only for  $n = 0$ , so we are not justified in assuming that  $n \geq 1$  when we try to prove the statement for  $n + 1$  in the inductive step. And of course the proposition first breaks precisely at  $n = 1$ . ■

**Problem 4.**

Define the *potential*,  $p(S)$ , of a stack of blocks,  $S$ , to be  $k(k-1)/2$  where  $k$  is the number of blocks in  $S$ . Define the potential,  $p(A)$ , of a set of stacks,  $A$ , to be the sum of the potentials of the stacks in  $A$ .

Generalize Theorem 6.2.2 about scores in the stacking game to show that for any set of stacks,  $A$ , if a sequence of moves starting with  $A$  leads to another set of stacks,  $B$ , then  $p(A) \geq p(B)$ , and the score for this sequence of moves is  $p(A) - p(B)$ .

*Hint:* Try induction on the number of moves to get from  $A$  to  $B$ .

**Solution.** *Proof.* The proof is by ordinary induction on the number of moves,  $n$ . The induction hypothesis will be

$P(n)::=$  If  $n$  moves from a set of stacks,  $A$ , leads to a set  $B$  of stacks, then  $p(A) \geq p(B)$  and the score for these  $n$  moves is  $p(A) - p(B)$ .

**Base case:** ( $n = 0$ ) This means no moves have been made and  $B = A$ , so it's obvious that  $P(0)$  holds.

**Inductive step:** Assume that  $P(n)$  is true for some  $n \in \mathbb{N}$ , and suppose  $A$  leads to  $B$  in  $n+1$  moves. This means that  $A$  leads to some set of stacks,  $A_1$ , and  $A_1$  leads to  $B$  in  $n$  steps. So the inductive hypothesis  $P(n)$  implies that  $p(A_1) \geq p(B)$  and the score for going from  $A_1$  to  $B$  is  $p(A_1) - p(B)$ .

So all we have to do is show that the score for the single move from  $A$  to  $A_1$  is  $p(A) - p(A_1) > 0$ . The only difference between  $A$  and  $A_1$  is that some stack  $S \in A$  of size  $k > 1$  splits into two stacks of sizes  $k_1, k_2 \geq 1$  where  $k = k_1 + k_2$ . The score for such a move is  $k_1 k_2$ . Also,

$$p(S) = \frac{(k_1 + k_2)((k_1 + k_2) + 1)}{2} = \frac{(k_1^2 + 2k_1 k_2 + k_2^2) + (k_1 + k_2)}{2},$$

and the potential of the two stack sets is the sum of their potentials, namely,

$$\frac{k_1(k_1 + 1) + k_2(k_2 + 1)}{2} = \frac{k_1^2 + k_2^2 + k_1 + k_2}{2},$$

So the difference between these potentials equals  $k_1 k_2 > 0$ , and this is indeed equal to the score of the move. ■

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