

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

# Induction

Albert R Meyer, February 22, 2010 lec 4M.1

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## The Idea of Induction

Color the integers  $\geq 0$

$0, 1, 2, 3, 4, 5, \dots$

I tell you,  $0$  is red, & any int next to a red integer is red, then you know that all the ints are red!

Albert R Meyer, February 22, 2010 lec 4M.2

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## The Idea of Induction

Color the integers  $\geq 0$

$0, 1, 2, 3, 4, 5, \dots$

I tell you,  $0$  is red, & any int next to a red integer is red, then you know that all the ints are red!

Albert R Meyer, February 22, 2010 lec 4M.3

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Induction Rule

$R(0), \forall n. R(n) \text{ IMPLIES } R(n+1)$ 


---

 $\forall n. R(n)$

Albert R Meyer, February 22, 2010 lec 4M.4

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Like Dominos...

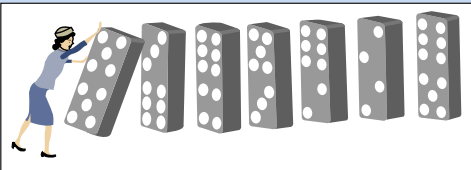


Image by MIT OpenCourseWare.

Albert R Meyer, February 22, 2010 lec 4M.5

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Example Induction Proof

Let's prove:

$$1+r+r^2+\dots+r^n = \frac{r^{(n+1)}-1}{r-1}$$

(for  $r \neq 1$ )

Albert R Meyer, February 22, 2010 lec 4M.6

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

### Example Induction Proof

Statements in magenta form a template for inductive proofs:

**Proof:** (by induction on  $n$ )

The induction hypothesis,  $P(n)$ , is:

$$1+r+r^2+\dots+r^n = \frac{r^{(n+1)}-1}{r-1}$$

(for  $r \neq 1$ )

Albert R Meyer, February 22, 2010 lec 4M.7

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

### Example Induction Proof

**Base Case ( $n = 0$ ):**

$$\underbrace{1+r+r^2+\dots+r^0}_1 = \frac{r^{0+1}-1}{r-1} = \frac{r-1}{r-1} = 1$$

OK!

Albert R Meyer, February 22, 2010 lec 4M.8

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

### Example Induction Proof

**Inductive Step:** Assume  $P(n)$  for some  $n \geq 0$  and prove  $P(n+1)$ :

$$1+r+r^2+\dots+r^{n+1} = \frac{r^{(n+1)+1}-1}{r-1}$$

Albert R Meyer, February 22, 2010 lec 4M.10

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

### Example Induction Proof

**Now from induction hypothesis  $P(n)$  we have**

$$1+r+r^2+\dots+r^n = \frac{r^{n+1}-1}{r-1}$$

so add  $r^{n+1}$  to both sides

Albert R Meyer, February 22, 2010 lec 4M.11

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

### Example Induction Proof

adding  $r^{n+1}$  to both sides,

$$(1+r+r^2+\dots+r^n)+r^{n+1} = \left(\frac{r^{n+1}-1}{r-1}\right)+r^{n+1}$$


**This proves  $P(n+1)$  completing the proof by induction.**

$$= \frac{r^{n+1}-1+r^{n+1}(r-1)}{r-1} = \frac{r^{(n+1)+1}-1}{r-1}$$

Albert R Meyer, February 22, 2010 lec 4M.12

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

### The MIT Stata Center



Copyright © 2003, 2004, 2005 Norman Walsh.  
This work is licensed under a Creative Commons license.

Albert R Meyer, February 22, 2010 lec 4M.15

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Design Mockup: Stata Lobby

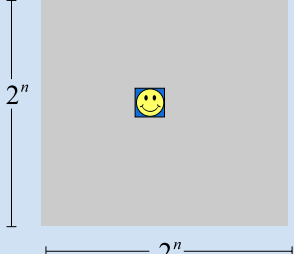
Image removed due to copyright restrictions.

Albert R Meyer, February 22, 2010 lec 4M.16

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Mockup: Plaza Outside Stata

Goal: Tile the plaza, except for 1x1 square in the middle for Bill.




Albert R Meyer, February 22, 2010 lec 4M.17

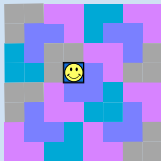
6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Plaza Outside Stata

Gehry specifies L-shaped tiles covering three squares:



For example, for 8 x 8 plaza might tile for Bill this way:



Albert R Meyer, February 22, 2010 lec 4M.18

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2


## Plaza Outside Stata

**Theorem:** For any  $2^n \times 2^n$  plaza, we can make Bill and Frank happy.

**Proof:** (by induction on  $n$ )

$P(n) ::=$  can tile  $2^n \times 2^n$  with Bill in middle.

**Base case:** ( $n=0$ )

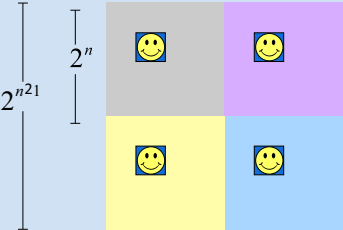
 (no tiles needed)

Albert R Meyer, February 22, 2010 lec 4M.19

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Plaza Outside Stata

**Induction step:** assume can tile  $2^n \times 2^n$ , prove can tile  $2^{n+1} \times 2^{n+1}$ .

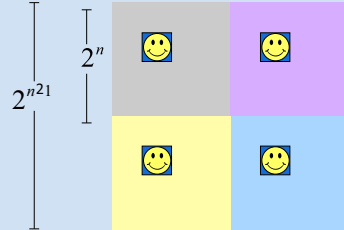


Albert R Meyer, February 22, 2010 lec 4M.20

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Plaza Outside Stata

Now what?



Albert R Meyer, February 22, 2010 lec 4M.21

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Plaza Outside Stata

The fix:

prove something stronger  
--that we can always  
find a tiling with Bill  
in any square.

Albert R Meyer, February 22, 2010 lec 4M.22

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Plaza Theorem

Theorem: For any  $2^n \times 2^n$  plaza, we can make Bill and Frank happy.

Proof: (by induction on  $n$ )

REVISED induction hypothesis  $P(n) ::=$   
can tile  $2^n \times 2^n$  with Bill anywhere

Base case: ( $n=0$ ) as before

Albert R Meyer, February 22, 2010 lec 4M.23

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Plaza Proof

Induction step:

Assume we can get Bill anywhere in  $2^n \times 2^n$ .

Prove we can get Bill anywhere in  $2^{n+1} \times 2^{n+1}$ .

Albert R Meyer, February 22, 2010 lec 4M.24

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Plaza Proof

Now group the squares together, and fill the center Bill's with a tile.

Done!

Albert R Meyer, February 22, 2010 lec 4M.25

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Recursive Procedure

Note: The induction proof implicitly defines a recursive procedure for tiling with Bill anywhere.

Albert R Meyer, February 22, 2010 lec 4M.26

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## A False Proof

Theorem: All horses are the same color.

Proof: (by induction on  $n$ )

Induction hypothesis:  
 $P(n) ::=$  any set of  $n$  horses have the same color

Base case ( $n=0$ ):  
No horses so vacuously true!

Albert R Meyer, February 22, 2010 lec 4M.27

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## A False Proof

(Inductive case)  
Assume any  $n$  horses have the same color.  
Prove that any  $n+1$  horses have the same color.

Albert R Meyer, February 22, 2010 lec 4M.28

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## A False Proof

(Inductive case)  
Assume any  $n$  horses have the same color.  
Prove that any  $n+1$  horses have the same color.

Albert R Meyer, February 22, 2010 lec 4M.29

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## A False Proof

(Inductive case)  
Assume any  $n$  horses have the same color.  
Prove that any  $n+1$  horses have the same color.

Albert R Meyer, February 22, 2010 lec 4M.30

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## A False Proof

### What's wrong?

Proof that  $P(n) \rightarrow P(n+1)$  is **wrong**  
if  $n = 1$ , because the two horse groups **do not overlap**.

Albert R Meyer, February 22, 2010 lec 4M.31

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## A False Proof

Proof that  $P(n) \rightarrow P(n+1)$  is **wrong**  
if  $n = 1$ , because the two horse groups **do not overlap**.

(But proof works for all  $n \neq 1$ )

Albert R Meyer, February 22, 2010 lec 4M.32

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

### Strong Induction

Prove  $P(0)$ . Then prove  $P(n+1)$   
assuming all of  
 $P(0), P(1), \dots, P(n)$   
(instead of just  $P(n)$ ).

Conclude  $\forall m. P(m)$

Albert R Meyer, February 22, 2010 lec 4M.33

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Unstacking game

Start: a stack of boxes

Move: split any stack into two of sizes  $a, b > 0$

Scoring:  $ab$  points

Keep moving: until stuck

Overall score: sum of move scores

Albert R Meyer, February 22, 2010 lec 4M.34

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Analyzing the Stacking Game

Claim: Every way of unstacking  $n$  blocks gives the same score:

$$(n-1) + (n-2) + \dots + 1 = \frac{n(n-1)}{2}$$

Albert R Meyer, February 22, 2010 lec 4M.35

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Analyzing the Game

Claim: Starting with size  $n$  stack, final score will be

$$\frac{n(n-1)}{2}$$

Proof: by Induction with Claim( $n$ ) as hypothesis

Albert R Meyer, February 22, 2010 lec 4M.36

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Proving the Claim by Induction

Base case  $n = 0$ :

$$\text{score} = 0 = \frac{0(0-1)}{2}$$

Claim(0) is

Albert R Meyer, February 22, 2010 lec 4M.37

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Proving the Claim by Induction

**Inductive step.** assume for stacks  $\leq n$ , and prove  $C(n+1)$ :

$$(n+1)\text{-stack score} = \frac{(n+1)n}{2}$$

Albert R Meyer, February 22, 2010 lec 4M.38

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

## Proving the Claim by Induction

**Inductive step.**

Case  $n+1 = 1$ . verify for 1-stack:

$$\text{score} = 0 = \frac{1(1-1)}{2}$$


$C(1)$  is

Albert R Meyer, February 22, 2010 lec 4M.39

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proving the Claim by Induction


**Inductive step.**  
 Case  $n+1 > 1$ . Split into an  
 $a$ -stack and  $b$ -stack,  
 where  $a + b = n+1$ .  
 $(a + b)$ -stack score =  $ab +$   
 $a$ -stack score +  $b$ -stack score

 Albert R Meyer, February 22, 2010 lec 4M.40

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proving the Claim by Induction


by **strong** induction:  
 $a$ -stack score =  $\frac{a(a-1)}{2}$   
 $b$ -stack score =  $\frac{b(b-1)}{2}$

 Albert R Meyer, February 22, 2010 lec 4M.41

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Proving the Claim by Induction

total  $(a + b)$ -stack score =  
 $ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} =$   
 $\frac{(a+b)((a+b)-1)}{2} = \frac{(n+1)n}{2}$   
 so  $C(n+1)$  is **o.k.**  
**We're done!**


 Albert R Meyer, February 22, 2010 lec 4M.42

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Team Problems

# Problems

## 1-4

 Albert R Meyer, February 22, 2010 lec 4M.43

MIT OpenCourseWare  
<http://ocw.mit.edu>

6.042J / 18.062J Mathematics for Computer Science  
Spring 2010

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.