Induction

The Idea of Induction

Color the integers ≥ 0

0, 1, 2, 3, 4, 5, …

I tell you, 0 is red, & any int next to a red integer is red, then you know that all the ints are red!

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Induction Rule

$$\frac{R(0),\ \forall n.\, R(n) \text{ IMPLIES } R(n+1)}{\forall n.\, R(n)}$$

Like Dominos…

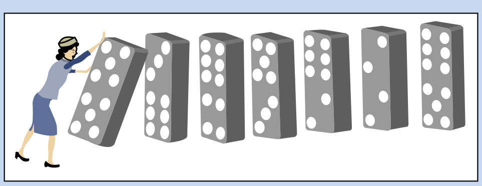


Image by MIT OpenCourseWare.

Example Induction Proof

Let's prove:

$$1+r+r^2+\cdots+r^n = \frac{r^{(n+1)}-1}{r-1}$$

(for $r \neq 1$)

Example Induction Proof

Statements in magenta form a **template for inductive proofs:**

Proof: (by induction on n)

The induction hypothesis, $P(n)$, is:

$$1+r+r^2+\cdots+r^n = \frac{r^{(n+1)}-1}{r-1}$$

(for $r \neq 1$)

Example Induction Proof

Base Case ($n = 0$):

$$\underbrace{1+r+r^2+\cdots+r^0}_{1} \overset{?}{=} \frac{r^{0+1}-1}{r-1} = \frac{r-1}{r-1} = 1$$

OK!

Example Induction Proof

Inductive Step: Assume $P(n)$ for some $n \geq 0$ and prove $P(n+1)$:

$$1+r+r^2+\cdots+r^{n+1} = \frac{r^{(n+1)+1}-1}{r-1}$$

Example Induction Proof

Now from induction hypothesis $P(n)$ we have

$$1+r+r^2+\cdots+r^n = \frac{r^{n+1}-1}{r-1}$$

so add r^{n+1} to both sides

Example Induction Proof

adding r^{n+1} to both sides,

$$\left(1+r+r^2+\cdots+r^n\right)+r^{n+1} = \left(\frac{r^{n+1}-1}{r-1}\right)+r^{n+1}$$

$$= \frac{r^{n+1}-1+r^{n+1}(r-1)}{r-1}$$

$$= \frac{r^{(n+1)+1}-1}{r-1}$$

This proves $P(n+1)$ completing the proof by induction.

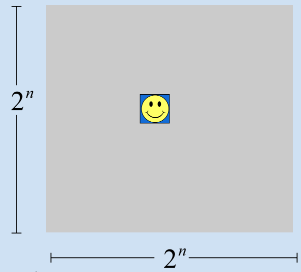
The MIT Stata Center

Design Mockup: Stata Lobby

Image removed due to copyright restrictions.

Mockup: Plaza Outside Stata

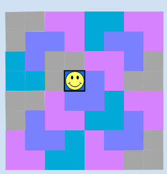
Goal: Tile the plaza, except for 1×1 square in the middle for Bill.



Plaza Outside Stata

Gehry specifies L-shaped tiles covering three squares: 

For example, for 8 x 8 plaza might tile for Bill this way:



Plaza Outside Stata

Theorem: For any $2^n \times 2^n$ plaza, we can make Bill and Frank happy.

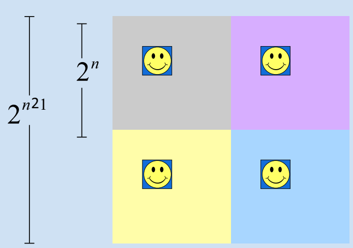
Proof: (by induction on n)
P(n) ::= can tile $2^n \times 2^n$ with Bill in middle.

Base case: (n=0)

 (no tiles needed)

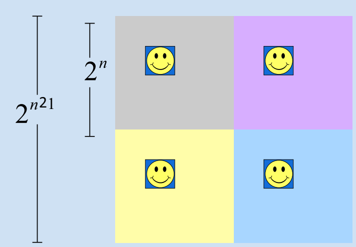
Plaza Outside Stata

Induction step: assume can tile $2^n \times 2^n$, prove can tile $2^{n+1} \times 2^{n+1}$.



Plaza Outside Stata

Now what?



Plaza Outside Stata

The fix:

prove something stronger --that we can always find a tiling with Bill in any square.

Plaza Theorem

Theorem: For any $2^n \times 2^n$ plaza, we can make Bill and Frank happy.

Proof: (by induction on n)

REVISED induction hypothesis P(n) ::=

can tile $2^n \times 2^n$ with Bill anywhere

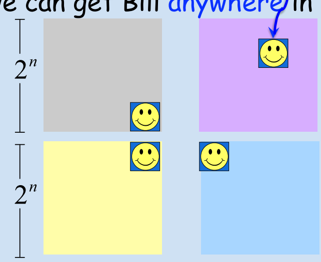
Base case: (n=0) as before

Plaza Proof

Induction step:

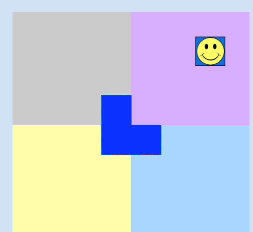
Assume we can get Bill anywhere in $2^n \times 2^n$.

Prove we can get Bill anywhere in $2^{n+1} \times 2^{n+1}$.



Plaza Proof

Now group the squares together, and fill the center Bill's with a tile.



Done!

Recursive Procedure

Note: The induction proof implicitly defines a

recursive procedure

for tiling with Bill anywhere.

A False Proof

Theorem: All horses are the same color.

Proof: (by induction on n)

Induction hypothesis:

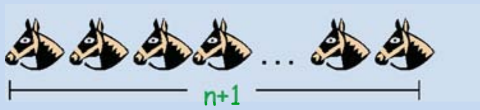
P(n) ::= any set of n horses have the same color

Base case (n=0):

No horses so vacuously true!

A False Proof

(Inductive case)
Assume any n horses have the same color.
Prove that any n+1 horses have the same color.



A False Proof

(Inductive case)
Assume any n horses have the same color.
Prove that any n+1 horses have the same color.

A False Proof

(Inductive case)
Assume any n horses have the same color.
Prove that any n+1 horses have the same color.

A False Proof

What's wrong?

Proof that P(n) → P(n+1) is wrong if n = 1, because the two horse groups do not overlap.

A False Proof

Proof that P(n) → P(n+1) is wrong if n = 1, because the two horse groups do not overlap.

(But proof works for all n ≠ 1)

Strong Induction

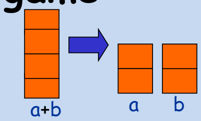
Prove P(0). Then prove P(n+1) assuming all of

P(0), P(1), …, P(n)

(instead of just P(n)).

Conclude ∀m.P(m)

Unstacking game



Start: a stack of boxes

Move: split any stack into two of sizes a,b>0

Scoring: ab points

Keep moving: until stuck

Overall score: sum of move scores

Analyzing the Stacking Game

Claim: Every way of unstacking n blocks gives the same score:

$$(n-1)+(n-2)+\cdots+1 = \frac{n(n-1)}{2}$$

Analyzing the Game

Claim: Starting with size n stack, final score will be

$$\frac{n(n-1)}{2}$$

Proof: by Induction with Claim(n) as hypothesis

Proving the Claim by Induction

Base case n = 0:

$$\text{score} = 0 = \frac{0(0-1)}{2}$$

Claim(0) is O.K.

Proving the Claim by Induction

Inductive step. assume for stacks ≤ n, and prove *C*(n+1):

$$(n+1)\text{-stack score} = \frac{(n+1)n}{2}$$

Proving the Claim by Induction

Inductive step.

Case n+1 = 1. verify for 1-stack:

$$\text{score} = 0 = \frac{1(1-1)}{2}$$

C(1) is O.K.

Proving the Claim by Induction

Inductive step.

Case $n+1 > 1$. Split into an a-stack and b-stack, where $a + b = n + 1$.

$(a + b)$-stack score = ab + a-stack score + b-stack score

Proving the Claim by Induction

by strong induction:

$$a\text{-stack score} = \frac{a(a-1)}{2}$$

$$b\text{-stack score} = \frac{b(b-1)}{2}$$

Proving the Claim by Induction

total $(a + b)$-stack score =

$$ab + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} =$$

$$\frac{(a+b)((a+b)-1)}{2} = \frac{(n+1)n}{2}$$

so $C(n+1)$ is O.K.

We're done!

Team Problems

Problems 1–4

6.042J / 18.062J Mathematics for Computer Science
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