Mathematics for Computer Science
MIT 6.042J/18.062J

Set Theory

Albert R Meyer, February 19, 2010 lec 3F.1

Axioms

Equality

$\forall x[x \in y \leftrightarrow x \in z] \rightarrow y = z$

Power set

$\forall x \exists p \forall s. s \subseteq x \leftrightarrow s \in p$

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Russell's Paradox

Let $W ::= \{s \in \text{Sets} \mid s \notin s\}$

so $[s \in W \text{ IFF } s \notin s]$

Now let s be W, and reach a contradiction:

$[W \in W \text{ IFF } W \notin W]$

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Disaster: Math is broken!

I am the Pope,
Pigs fly,
and verified programs crash...

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...but paradox is buggy

Assumes that W is a set!

$[s \in W \text{ IFF } s \notin s]$

for all sets s
...can only substitute W for s if W is a set

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...but paradox is buggy

Assumes that W is a set!

We can avoid the paradox, if we deny that W is a set!
...which raises the key question: just which well-defined collections *are* sets?

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Zermelo-Frankel Set Theory

No simple answer, but the axioms of Zermelo-Frankel along with the Choice axiom (ZFC) do a pretty good job.

Zermelo-Frankel Set Theory

According to ZF, the elements of a set have to be "simpler" than the set itself. In particular,

no set is a member of itself.

Zermelo-Frankel Set Theory

This implies that

(1) the collection of all sets is not a set, and

(2) W equals the collection of all sets …which is why it's not a set

infinite sizes

Are infinite sets the "same size"?

NO, by Russell paradox variant:

Theorem: No surjective function from A to $\text{pow}(A)$, even for infinite A

no surjection from A to $\text{pow}(A)$

Pf by contradiction: suppose surj fcn $f: A \to \text{pow}(A)$. Let

$W ::= \{a \in A \mid a \notin f(a)\}$, so

$a \in W$ iff $a \notin f(a)$.

f a surj, so $W = f(a_0)$, some $a_0 \in A$.

So, $a_0 \in f(a_0)$ iff $a_0 \notin f(a_0)$. ∎

$\{0,1\}^\omega$ is uncountable

A is *countable* iff can be listed $a_0, a_1, a_2, \ldots$.

same as surj fcn: $\mathbb{N} \to A$

So $\{0,1\}^\omega$ is uncountable, because

$$\mathbb{N} \to \{0,1\}^\omega \to \text{pow}(\mathbb{N})$$

surj

surj bij

∎

Team Problems

Problems 1—3

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