

## Solutions to In-Class Problems Week 3, Tue.

### Problem 1.

Lemma 4.9.4. Let  $A$  be a set and  $b \notin A$ . If  $A$  is infinite, then there is a bijection from  $A \cup \{b\}$  to  $A$ .

*Proof.* Here's how to define the bijection: since  $A$  is infinite, it certainly has at least one element; call it  $a_0$ . But since  $A$  is infinite, it has at least two elements, and one of them must not be equal to  $a_0$ ; call this new element  $a_1$ . But since  $A$  is infinite, it has at least three elements, one of which must not equal  $a_0$  or  $a_1$ ; call this new element  $a_2$ . Continuing in the way, we conclude that there is an infinite sequence  $a_0, a_1, a_2, \dots, a_n, \dots$  of different elements of  $A$ . Now we can define a bijection  $f : A \cup \{b\} \rightarrow A$ :

$$\begin{aligned} f(b) &::= a_0, \\ f(a_n) &::= a_{n+1} && \text{for } n \in \mathbb{N}, \\ f(a) &::= a && \text{for } a \in A - \{b, a_0, a_1, \dots\}. \end{aligned}$$

■

(a) Several students felt the proof of Lemma 4.9.4 was worrisome, if not circular. What do you think?

**Solution.** There is no "solution" for this discussion problem, since it depends on what seems bothersome.

It may be bothersome that the proof asserts that  $f$  is bijection without spelling out a proof. But the bijection property really does follow directly from definition of  $f$ , so it shouldn't be much burden for a bothered reader to fill in such a proof.

Another possibly bothersome point is that the proof assumes that if a set is infinite, it must have more than  $n$  elements, for every nonnegative integer  $n$ . But really that's the definition of infinity: a set is finite iff it has  $n$  elements for some nonnegative integer,  $n$ , and a set is infinite iff it is *not* finite.

A possibly worrisome point is how you find an element  $a_{n+1} \in A$  given  $a_0, a_1, \dots, a_n$ . But you don't have to *find* a specific one: there must be an element in  $A - \{a_0, a_1, \dots, a_n\}$ —so just pick any one. Actually, the justification for this step is the set-theoretic Axiom of Choice described in the Notes chapter first-order logic, and some logicians do consider it worrisome. ■

(b) Use the proof of Lemma 4.9.4 to show that if  $A$  is an infinite set, then there is surjective function from  $A$  to  $\mathbb{N}$ , that is, every infinite set is "as big as" the set of nonnegative integers.

**Solution.** By the proof of Lemma 4.9.4, there is an infinite sequence  $a_0, a_1, a_2, \dots, a_n, \dots$  of different elements of  $A$ . Then we can define a surjective function  $f : A \rightarrow \mathbb{N}$  by defining

$$f(a) ::= \begin{cases} n, & \text{if } a = a_n, \\ \text{undefined,} & \text{otherwise.} \end{cases}$$

—A total surjective function is not required, but if you want one define  $f' : A \rightarrow \mathbb{N}$ , by

$$f'(a) ::= \begin{cases} n, & \text{if } a = a_n, \\ 0, & \text{otherwise.} \end{cases}$$

■

### Problem 2.

Let  $R : A \rightarrow B$  be a binary relation. Use an arrow counting argument to prove the following generalization of the Mapping Rule:

**Lemma.** *If  $R$  is a function, and  $X \subseteq A$ , then*

$$|X| \geq |XR|.$$

**Solution.** *Proof.* The proof is virtually a repeat of the proof in the Appendix for the first Mapping Rule.

Since  $R$  is a function, the number of arrows whose starting point is an element of  $X$  is at most the number of elements in  $X$ . That is,

$$|X| \geq \#\text{arrows from } X.$$

Also, each element of  $XR$  is, by definition, the endpoint of at least one arrow starting from  $X$ , so there must be at least as many arrows starting from  $X$  as the number of elements of  $XR$ . That is,

$$\#\text{arrows from } X \geq |XR|.$$

Combining these inequalities immediately implies that  $|X| \geq |XR|$ . ■

An alternative proof appeals to the original Mapping Rule:

*Proof.* Consider the relation  $R'$  whose domain is  $X$ , whose codomain is  $XR$ , and whose arrows are just the arrows of  $R$  that start from  $X$ . (These arrows necessarily end in  $XR$  by definition of  $XR$ .) Since  $R$  is a function,  $R'$  will be one too, and by definition of  $XR$ , the relation  $R'$  is a surjection. Hence the first Mapping Rule implies that  $|X| \geq |XR|$ . ■

### Problem 3.

Let  $A = \{a_0, a_1, \dots, a_{n-1}\}$  be a set of size  $n$ , and  $B = \{b_0, b_1, \dots, b_{m-1}\}$  a set of size  $m$ . Prove that  $|A \times B| = mn$  by defining a simple bijection from  $A \times B$  to the nonnegative integers from 0 to  $mn - 1$ .

**Solution.** A bijection  $f : A \times B \rightarrow \{0, 1, \dots, mn - 1\}$  can be defined by the rule

$$f(a_k, b_j) ::= jn + k.$$

■

#### Problem 4.

The rational numbers fill in all the spaces between the integers, so a first thought is that there must be more of them than the integers, but it's not true. In this problem you'll show that there are the same number of nonnegative rational as nonnegative integers. In short, the nonnegative rationals are countable.

(a) Describe a bijection between all the integers,  $\mathbb{Z}$ , and the nonnegative integers,  $\mathbb{N}$ .

**Solution.** One such bijection is defined by lining up all the integers and the nonnegative integers as follows:

$$\begin{array}{cccccccc} 0 & 1 & -1 & 2 & -2 & 3 & -3 & 4 & \dots \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \dots \end{array}$$

We can also define this bijection,  $f : \mathbb{Z} \rightarrow \mathbb{Z}^+$ , by a specification rule

$$f(n) = \begin{cases} -2n & \text{for } n \leq 0, \\ 2|n| - 1 & \text{for } n > 0. \end{cases}$$

■

(b) Define a bijection between the nonnegative integers and the set,  $\mathbb{N} \times \mathbb{N}$ , of all the ordered pairs of nonnegative integers:

$$\begin{array}{l} (0, 0), (0, 1), (0, 2), (0, 3), (0, 4), \dots \\ (1, 0), (1, 1), (1, 2), (1, 3), (1, 4), \dots \\ (2, 0), (2, 1), (2, 2), (2, 3), (2, 4), \dots \\ (3, 0), (3, 1), (3, 2), (3, 3), (3, 4), \dots \\ \vdots \end{array}$$

**Solution.** Line up all the pairs by following successive upper-right to lower-left diagonals along the top row.

That is, start with (0,0) which counts as an initial diagonal of length 1. Then follow the length 2 second diagonal (0,1), (1,0), then the length 3 third diagonal (0,2), (1,1), (2,0), then the length 4 fourth diagonal (0,3), (1,2), (2,1), (3,0), .... So the line up would be

$$\begin{array}{cccccccccc} (0, 0) & (0, 1) & (1, 0) & (0, 2) & (1, 1) & (2, 0) & (0, 3) & (1, 2) & (2, 1) & (3, 0) & \dots \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \dots \end{array}$$

It's interesting that this bijection from  $\mathbb{N} \times \mathbb{N}$  to  $\mathbb{N}$  has a simple formula: the pair  $(k, m)$  is the  $k$ th element on the diagonal consisting of the pairs whose sum is  $k + m$ . The total number of elements in all the preceding diagonals is

$$0 + 1 + 2 + \dots + (k + m) = (k + m + 1)(k + m)/2$$

so the pair  $(k, m)$  appears as the  $(k + m + 1)(k + m)/2 + k$ th element in the line up.

■

(c) Conclude that  $\mathbb{N}$  is the same size as the set,  $\mathbb{Q}$ , of all nonnegative rational numbers.

**Solution.** One way to line up the nonnegative rationals is to take the list of all pairs,  $(k, m)$ , of integers above and replace each remaining pair by the rational number  $k/m$ , skipping the pairs where  $m = 0$ :

0/1 0/2 1/1 1/2 0/3 1/2 2/1 0/3 1/2 2/1 ...

and, going from left to right, delete all the occurrences of numbers that are already in the list:

0 1 1/2 2 3 1/3 1/4....



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6.042J / 18.062J Mathematics for Computer Science  
Spring 2010

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