In-Class Problems Week 3, Tue.

Problem 1.

Lemma 4.9.4. Let A be a set and $b \notin A$. If A is infinite, then there is a bijection from $A \cup \{b\}$ to A.

Proof. Here's how to define the bijection: since A is infinite, it certainly has at least one element; call it a_0. But since A is infinite, it has at least two elements, and one of them must not be equal to a_0; call this new element a_1. But since A is infinite, it has at least three elements, one of which must not equal a_0 or a_1; call this new element a_2. Continuing in the way, we conclude that there is an infinite sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ of different elements of A. Now we can define a bijection $f : A \cup \{b\} \to A$:

$$\begin{aligned} f(b) &::= a_0, \\ f(a_n) &::= a_{n+1} && \text{for } n \in \mathbb{N}, \\ f(a) &::= a && \text{for } a \in A - \{b, a_0, a_1, \ldots\}. \end{aligned}$$

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(a) Several students felt the proof of Lemma 4.9.4 was worrisome, if not circular. What do you think?

(b) Use the proof of Lemma 4.9.4 to show that if A is an infinite set, then there is surjective function from A to $\mathbb{N}$, that is, every infinite set is "as big as" the set of nonnegative integers.

Problem 2.
Let $R : A \to B$ be a binary relation. Use an arrow counting argument to prove the following generalization of the Mapping Rule:

Lemma. *If R is a function, and $X \subseteq A$, then*

$$|X| \geq |XR|.$$

Problem 3.
Let $A = \{a_0, a_1, \ldots, a_{n-1}\}$ be a set of size n, and $B = \{b_0, b_1, \ldots, b_{m-1}\}$ a set of size m. Prove that $|A \times B| = mn$ by defining a simple bijection from $A \times B$ to the nonnegative integers from 0 to $mn - 1$.

Problem 4.
The rational numbers fill in all the spaces between the integers, so a first thought is that there must be more of them than the integers, but it's not true. In this problem you'll show that there are the same number of nonnegative rational as nonnegative integers. In short, the nonnegative rationals are countable.

(a) Describe a bijection between all the integers, $\mathbb{Z}$, and the nonnegative integers, $\mathbb{N}$.

(b) Define a bijection between the nonnegative integers and the set, $\mathbb{N} \times \mathbb{N}$, of all the ordered pairs of nonnegative integers:

$$\begin{array}{l}(0,0),(0,1),(0,2),(0,3),(0,4),\dots \\ (1,0),(1,1),(1,2),(1,3),(1,4),\dots \\ (2,0),(2,1),(2,2),(2,3),(2,4),\dots \\ (3.0),(3,1),(3,2),(3,3),(3,4),\dots \\ \quad\vdots \end{array}$$

(c) Conclude that $\mathbb{N}$ is the same size as the set, $\mathbb{Q}$, of all nonnegative rational numbers.

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