Massachusetts Institute of Technology
6.042J/18.062J, Spring '10: Mathematics for Computer Science
Prof. Albert R. Meyer

February 12
revised February 11, 2010, 172 minutes

Solutions to In-Class Problems Week 2, Fri.

Problem 1.
Set Formulas and Propositional Formulas.

(a) Verify that the propositional formula $(P \text{ AND } \text{NOT}(Q)) \text{ OR } (P \text{ AND } Q)$ is equivalent to P.

Solution. There is a simple verification by truth table with 4 rows which we omit.

There is also a simple cases argument: if Q is **T**, then the formula simplifies to $(P \text{ AND } \mathbf{F}) \text{ OR } (P \text{ AND } \mathbf{T})$ which further simplifies to $(\mathbf{F} \text{ OR } P)$ which is equivalent to P.

Otherwise, if Q is F, then the formula simplifies to $(P \text{ AND } \mathbf{T}) \text{ OR } (P \text{ AND } \mathbf{F})$ which is likewise equivalent to P. ■

(b) Use part (a) to prove that

$$A = (A - B) \cup (A \cap B)$$

for any sets, A, B, where

$$A - B ::= \{a \in A \mid a \notin B\}.$$

Solution. We need only show that the two sets have the same elements, that is x is in one set iff x is in the other set, for any x.

Let P be $x \in A$ and Q be $x \in B$. Then

$$\begin{aligned}
& x \in (A - B) \cup (A \cap B) \\
& \quad \text{iff} \quad x \in (A - B) \text{ OR } x \in (A \cap B) && \text{(by def of } \cup) \\
& \quad \text{iff} \quad (x \in A \text{ AND } \text{NOT}(x \in B)) \text{ OR } (x \in A \text{ AND } x \in B) && \text{(by def of } \cap \text{ and } -) \\
& \quad \text{iff} \quad (P \text{ AND } \text{NOT}(Q)) \text{ OR } (P \text{ AND } Q) && \text{(by def of } P \text{ and } Q) \\
& \quad \text{iff} \quad P && \text{(by part (a))} \\
& \quad \text{iff} \quad x \in A && \text{(by def of } P).
\end{aligned}$$

■

Problem 2.
Subset take-away[1] is a two player game involving a fixed finite set, A. Players alternately choose nonempty subsets of A with the conditions that a player may not choose

- the whole set A, or

[1]From Christenson & Tilford, *David Gale's Subset Takeaway Game, American Mathematical Monthly, Oct. 1997*

- any set containing a set that was named earlier.

The first player who is unable to move loses the game.

For example, if A is $\{1\}$, then there are no legal moves and the second player wins. If A is $\{1, 2\}$, then the only legal moves are $\{1\}$ and $\{2\}$. Each is a good reply to the other, and so once again the second player wins.

The first interesting case is when A has three elements. This time, if the first player picks a subset with one element, the second player picks the subset with the other two elements. If the first player picks a subset with two elements, the second player picks the subset whose sole member is the third element. Both cases produce positions equivalent to the starting position when A has two elements, and thus leads to a win for the second player.

Verify that when A has four elements, the second player still has a winning strategy.[2]

Solution. There are way too many cases to work out by hand if we tried to list all possible games. But the elements of A all behave the same, so we can cut to a small number of cases using the fact that permuting around the elements of A in any game yields another possible game. We can do this by not mentioning specific elements of A, but instead using the *variables* a, b, c, d whose values will be the four elements of A.

We consider two cases for the move of the Player 1 when the game starts:

1. Player 1 chooses a one element or a three element subset. Then Player 2 should choose the complement of Player one's choice. The game then becomes the same as playing the $n = 3$ game on the three element set chosen in this first round, where we know Player 2 has a winning strategy.

2. Player 1 chooses a subset of 2 elements. Let a, b be these elements, that is, the first move is $\{a, b\}$. Player 2 should choose the complement, $\{c, d\}$, of Player 1's choice. We then have the following subcases:

   (a) Player 1's second move is a one element subset, $\{a\}$. Player 2 should choose $\{b\}$. The game is then reduced to the two element game on $\{c, d\}$ where Player 2 has a winning strategy.

   (b) Player 1's second move is a two element subset, $\{a, c\}$. Player 2 should choose its complement, $\{b, d\}$. This leads to two subsubcases:

      i. Player 1's third move is one of the remaining sets of size two, $\{a, d\}$. Player 2 should choose its complement, $\{b, c\}$. The remaining possible moves are the four sets of size 1, where the Player 2 clearly wins after two more rounds.

      ii. Player 1's third move is a one element set, $\{a\}$. Player 2 should choose $\{b\}$. The game is then reduced to the case two element game on $\{c, d\}$ where Player 2 has a winning strategy.

So in all cases, Player 2 has a winning strategy in the Gale game for $n = 4$. ■

[2] David Gale worked out some of the properties of this game and conjectured that the second player wins the game for any set A. This remains an open problem.

Problem 3.
Define a *surjection relation*, surj, on sets by the rule

Definition. $A \text{ surj } B$ iff there is a surjective **function** from A to B.

Define the *injection relation*, inj, on sets by the rule

Definition. $A \text{ inj } B$ iff there is a total injective *relation* from A to B.

(a) Prove that if $A \text{ surj } B$ and $B \text{ surj } C$, then $A \text{ surj } C$.

Solution. By definition of surj, there are surjective functions, $F : A \to B$ and $G : B \to C$.

Let $H ::= G \circ F$ be the function equal to the composition of G and F, that is

$$H(a) ::= G(F(a)).$$

We show that H is surjective, which will complete the proof. So suppose $c \in C$. Then since G is a surjection, $c = G(b)$ for some $b \in B$. Likewise, $b = F(a)$ for some $a \in A$. Hence $c = G(F(a)) = H(a)$, proving that c is in the range of H, as required. ■

(b) Explain why $A \text{ surj } B$ iff $B \text{ inj } A$.

Solution. *Proof.* (right to left): By definition of inj, there is a total injective relation, $R : B \to A$. But this implies that R^{-1} is a surjective function from A to B.

(left to right): By definition of surj, there is a surjective function, $F : A \to B$. But this implies that F^{-1} is a total injective relation from A to B. ■

(c) Conclude from (a) and (b) that if $A \text{ inj } B$ and $B \text{ inj } C$, then $A \text{ inj } C$.

Solution. From (b) and (a) we have that if $C \text{ inj } B$ and $B \text{ inj } A$, then $C \text{ inj } A$, so just switch the names A and C. ■