In-Class Problems Week 2, Fri.

Problem 1.
Set Formulas and Propositional Formulas.

(a) Verify that the propositional formula $(P \text{ AND } \text{NOT}(Q)) \text{ OR } (P \text{ AND } Q)$ is equivalent to P.

(b) Use part (a) to prove that

$$A = (A - B) \cup (A \cap B)$$

for any sets, A, B, where

$$A - B ::= \{a \in A \mid a \notin B\}.$$

Problem 2.
Subset take-away[1] is a two player game involving a fixed finite set, A. Players alternately choose nonempty subsets of A with the conditions that a player may not choose

- the whole set A, or
- any set containing a set that was named earlier.

The first player who is unable to move loses the game.

For example, if A is $\{1\}$, then there are no legal moves and the second player wins. If A is $\{1, 2\}$, then the only legal moves are $\{1\}$ and $\{2\}$. Each is a good reply to the other, and so once again the second player wins.

The first interesting case is when A has three elements. This time, if the first player picks a subset with one element, the second player picks the subset with the other two elements. If the first player picks a subset with two elements, the second player picks the subset whose sole member is the third element. Both cases produce positions equivalent to the starting position when A has two elements, and thus leads to a win for the second player.

Verify that when A has four elements, the second player still has a winning strategy.[2]

Problem 3.
Define a *surjection relation*, surj, on sets by the rule

Definition. A surj B iff there is a surjective **function** from A to B.

[1] From Christenson & Tilford, *David Gale's Subset Takeaway Game, American Mathematical Monthly, Oct. 1997*

[2] David Gale worked out some of the properties of this game and conjectured that the second player wins the game for any set A. This remains an open problem.

Define the *injection relation*, inj, on sets by the rule

Definition. $A \text{ inj } B$ iff there is a total injective *relation* from A to B.

(a) Prove that if $A \text{ surj } B$ and $B \text{ surj } C$, then $A \text{ surj } C$.

(b) Explain why $A \text{ surj } B$ iff $B \text{ inj } A$.

(c) Conclude from (a) and (b) that if $A \text{ inj } B$ and $B \text{ inj } C$, then $A \text{ inj } C$.