Mathematics for Computer Science

MIT 6.042J/18.062J

Sets & Functions

What is a Set?

Informally:

A ***set*** is a collection of mathematical objects, with the collection treated as a single mathematical object.

(This is circular of course: what's a *collection*?)

Some sets

real numbers, $\mathbb{R}$

complex numbers, $\mathbb{C}$

integers, $\mathbb{Z}$

empty set, $\varnothing$

set of all subsets of integers, pow($\mathbb{Z}$)

the power set

Some sets

{7, "Albert R.", π/2, T}

A set with 4 elements: two numbers, a string, and a Boolean. Same as

{T, "Albert R.", 7, π/2}

-- order doesn't matter

Membership

x is a member of A: $x \in A$

π/2 ∈ {7, "Albert R.", π/2, T}

π/3 ∉ {7, "Albert R.", π/2, T}

14/2 ∈ {7, "Albert R.", π/2, T}

Synonyms for Membership

$x \in A$

x is an element of A

x is in A

Examples:

$7 \in \mathbb{Z}$, $2/3 \notin \mathbb{Z}$, $\mathbb{Z} \in \text{pow}(\mathbb{R})$

In or Not In

An element is in or not in a set:

$\{7, \pi/2, 7\}$ is same as $\{7, \pi/2\}$

(No notion of being in the set more than once)

Subset ($\subseteq$)

$A \subseteq B$ A is a subset of B

A is contained in B

Every element of A is also an element of B:

$\forall x\ [x \in A \text{ IMPLIES } x \in B]$

Subset

examples:

$\mathbb{Z} \subseteq \mathbb{R}$, $\mathbb{R} \subseteq \mathbb{C}$, $\{3\} \subseteq \{5,7,3\}$

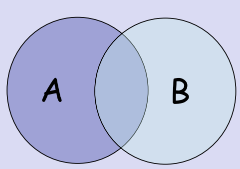
$A \subseteq A$, $\varnothing \subseteq$ every set

$\varnothing \subseteq$ everything

def: $\varnothing \subseteq B$

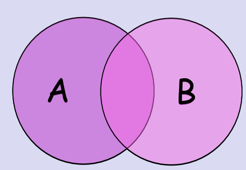
$\forall x\ [\underbrace{x \in \varnothing \text{ IMPLIES } x \in B}_{\text{true}}]$

New sets from old



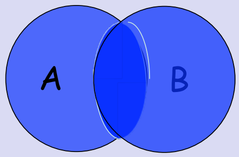
Venn Diagram for 2 Sets

union



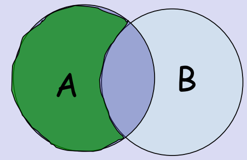
$A \cup B ::= \{x \mid x \in A \text{ OR } x \in B\}$

intersection



$$A \cap B ::= \{x \mid x \in A \text{ AND } x \in B\}$$

set difference



$$A - B ::= \{x \mid x \in A \text{ AND } x \notin B\}$$

A set-theoretic equality

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

proof: Show these have the same elements, namely,

$x \in$ Left Hand Set iff $x \in$ RHS

for all x.

A set-theoretic equality

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

proof uses fact from last time:

P OR (Q AND R) equiv

(P OR Q) AND (P OR R)

A set-theoretic equality

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

proof: $x \in A \cup (B \cap C)$ iff

$x \in A$ OR $x \in (B \cap C)$ (def of $\cup$) iff

$x \in A$ OR ($x \in B$ AND $x \in C$) (def $\cap$) iff

($x \in A$ OR $x \in B$) AND ($x \in A$ OR $x \in C$)

(by the equivalence)

A set-theoretic equality

proof:

($x \in A$ OR $x \in B$) AND ($x \in A$ OR $x \in C$) iff

($x \in A \cup B$) AND ($x \in A \cup C$) (def $\cup$) iff

$x \in (A \cup B) \cap (A \cup C)$ (def $\cap$).

QED

Relations & Functions

"is taking subject" relation

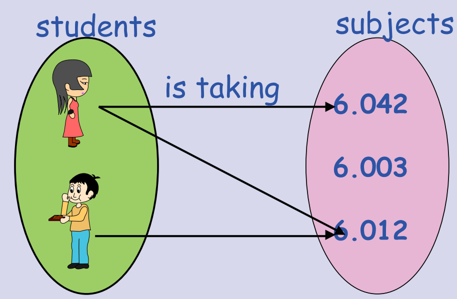
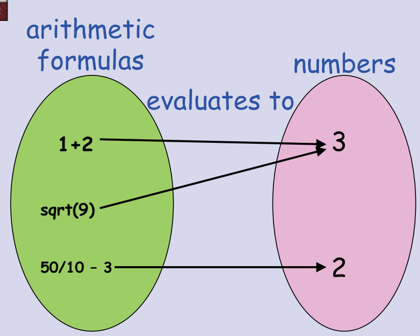
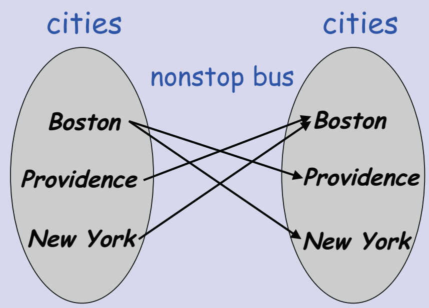


Image by MIT OpenCourseWare.

formula "evaluation" relation



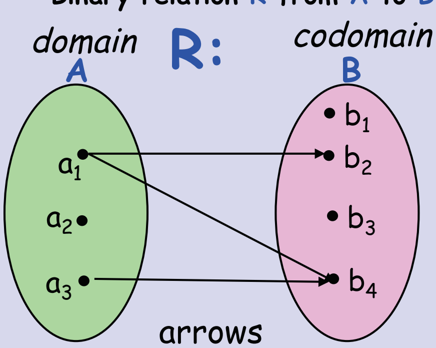
"nonstop bus trip" relation

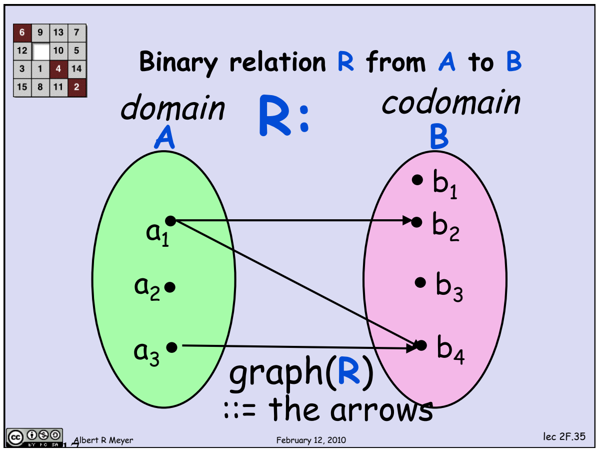


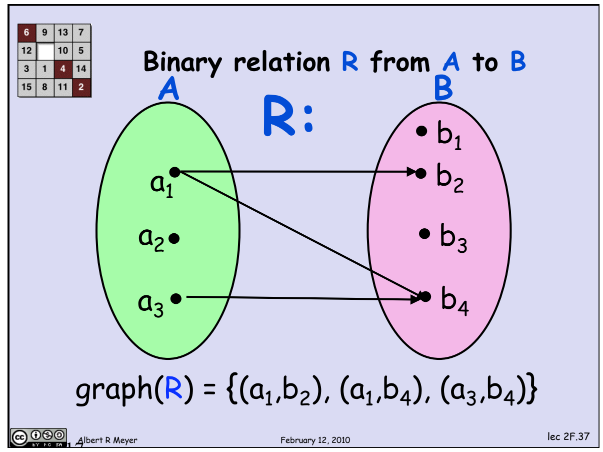
Binary relations

A binary relation, R, from a set A to a set B associates of elements of A with elements of B.

Binary relation R from A to B




Binary relation R from A to B
domain
R:
codomain
A
B
a_1
a_2
a_3
b_1
b_2
b_3
b_4
graph(R)
::= the arrows
Albert R Meyer
February 12, 2010
lec 2F.35


Binary relation R from A to B
A
R:
B
a_1
a_2
a_3
b_1
b_2
b_3
b_4
graph(R) = $\{(a_1,b_2), (a_1,b_4), (a_3,b_4)\}$
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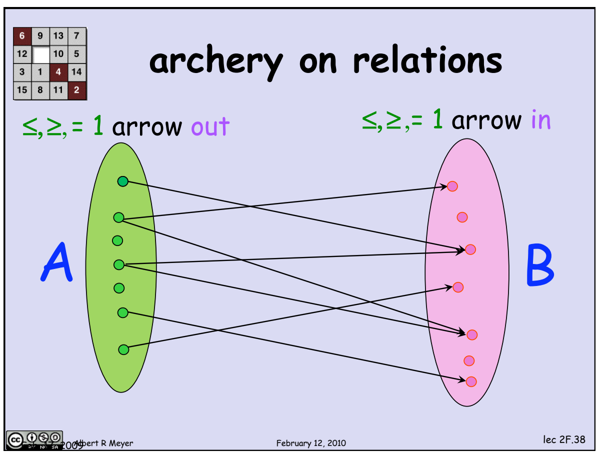
archery on relations
≤,≥,= 1 arrow out
≤,≥,= 1 arrow in
A
B
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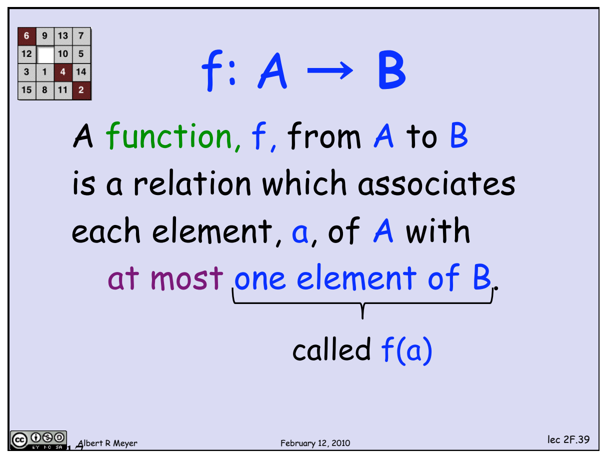
lec 2F.38


f: A → B
A function, f, from A to B
is a relation which associates
each element, a, of A with
at most one element of B.
called f(a)
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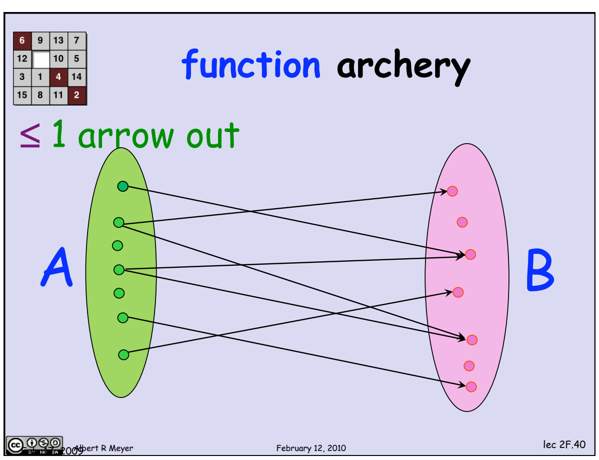
function archery
≤ 1 arrow out
A
B
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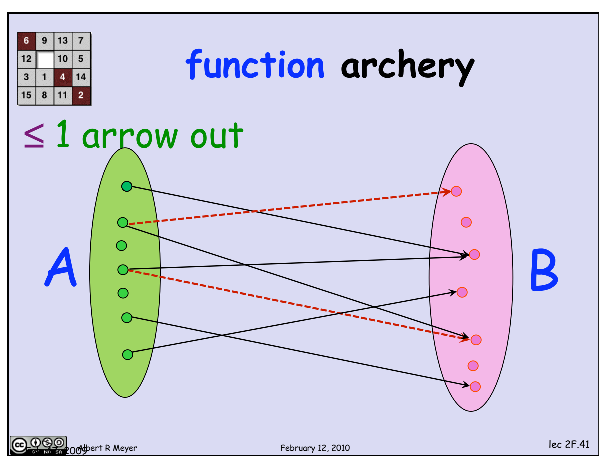
lec 2F.40


function archery
≤ 1 arrow out
A
B
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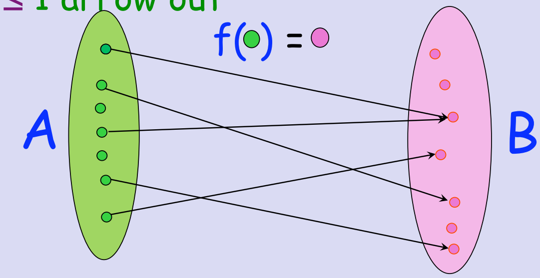
function archery

≤ 1 arrow out

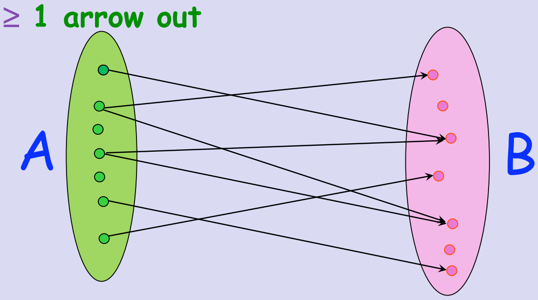


total relations

R:A→B is total iff every element of A is associated with at least one element of B

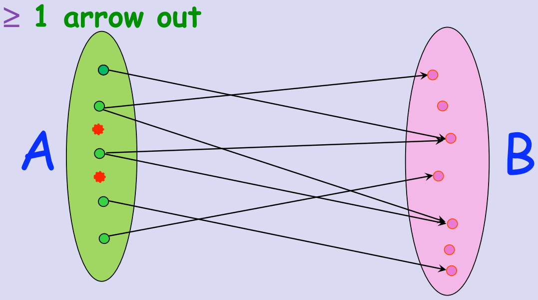
total relation archery

≥ 1 arrow out



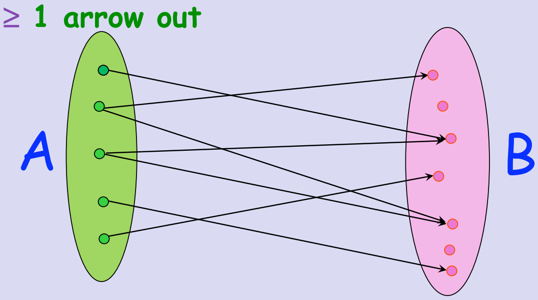
total relation archery

≥ 1 arrow out



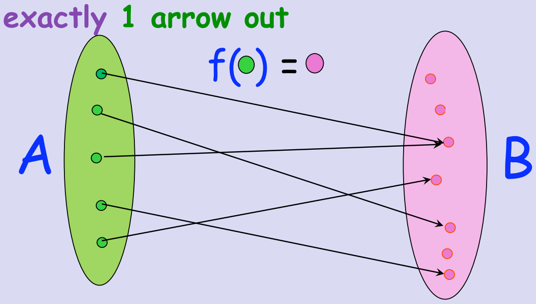
total relation archery

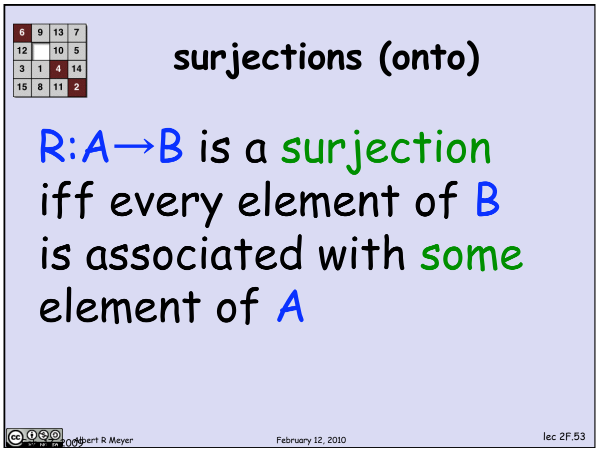
≥ 1 arrow out



total & function archery

exactly 1 arrow out




surjections (onto)
R:A→B is a surjection
iff every element of B
is associated with some
element of A
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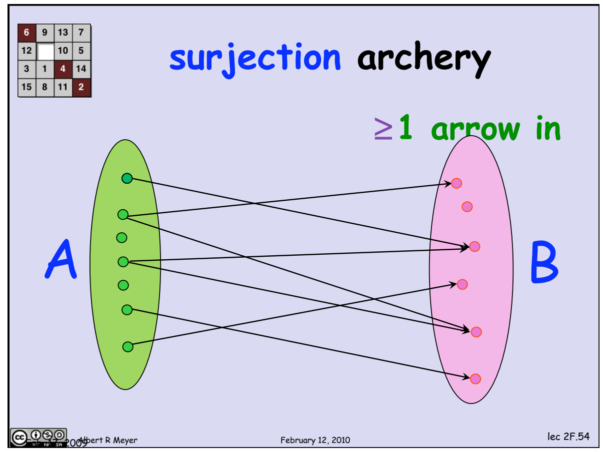
surjection archery
≥1 arrow in
A
B
Albert R Meyer
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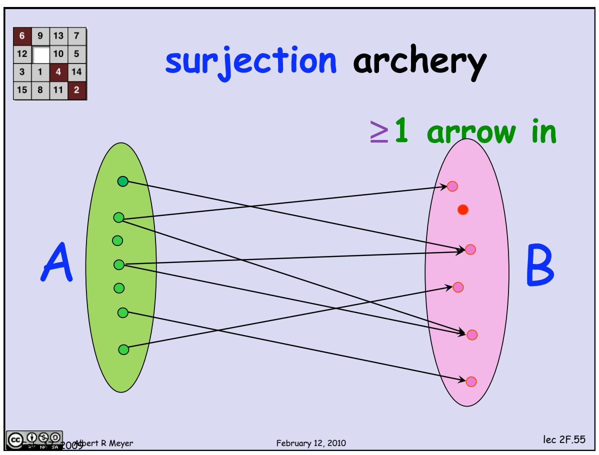
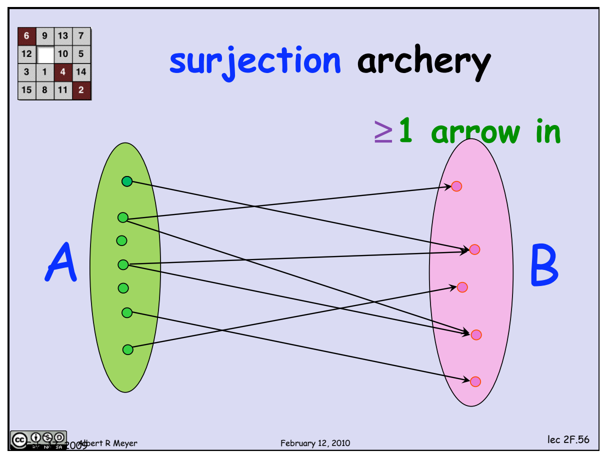
lec 2F.54


surjection archery
≥1 arrow in
A
B
Albert R Meyer
February 12, 2010
lec 2F.55


surjection archery
≥1 arrow in
A
B
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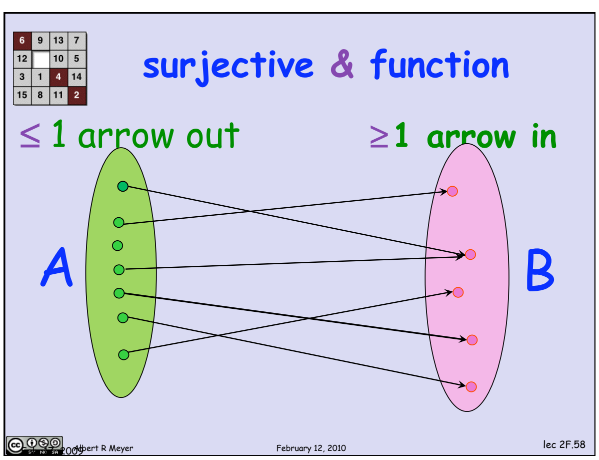
lec 2F.56


surjective & function
≤ 1 arrow out
≥1 arrow in
A
B
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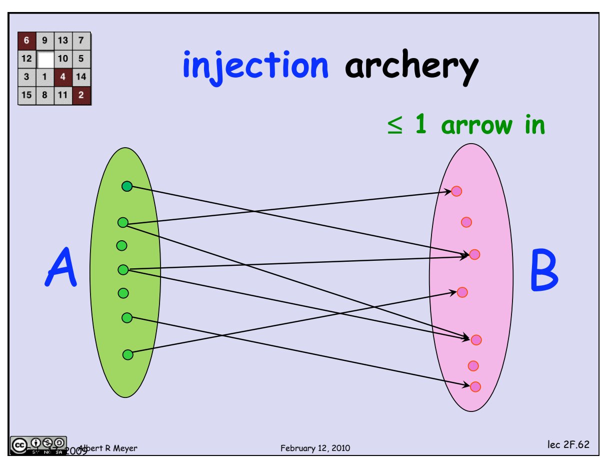
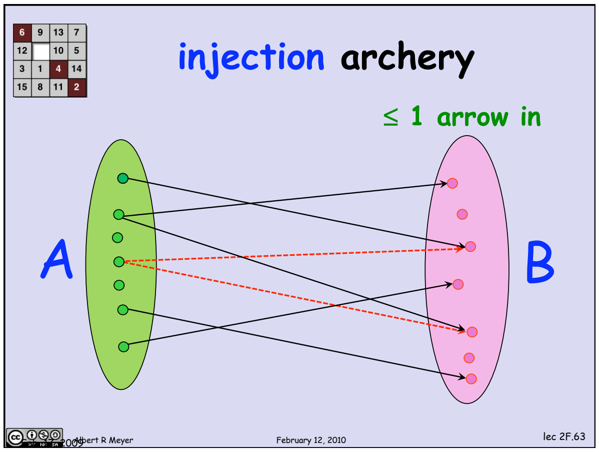
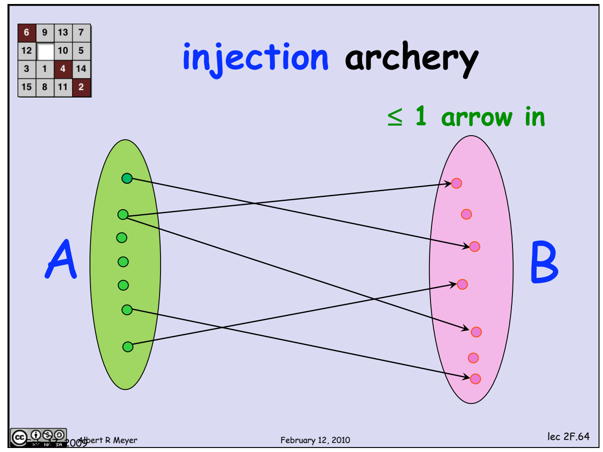
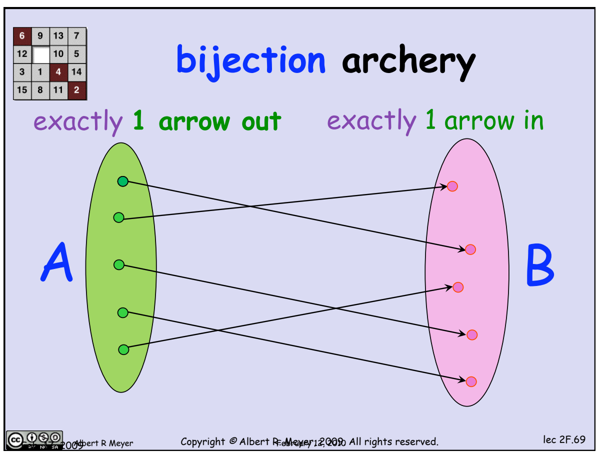
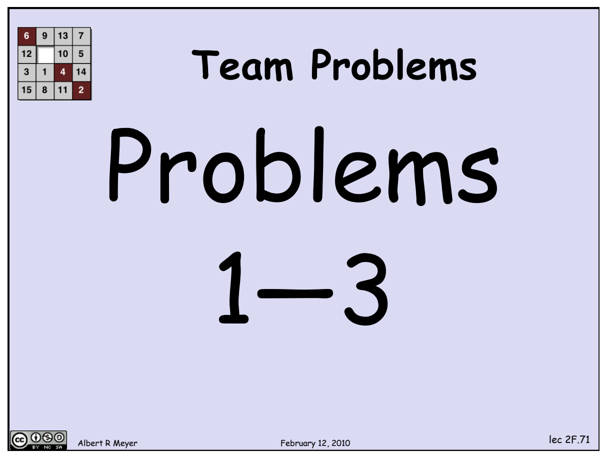
February 12, 2010
lec 2F.58


injection archery
≤ 1 arrow in
A
B
Albert R Meyer
February 12, 2010
lec 2F.62


injection archery
≤ 1 arrow in
A
B
Albert R Meyer
February 12, 2010
lec 2F.63


injection archery
≤ 1 arrow in
A
B
Albert R Meyer
February 12, 2010
lec 2F.64


bijection archery
exactly 1 arrow out
exactly 1 arrow in
A
B
lec 2F.69


Team Problems
Problems
1—3
Albert R Meyer
February 12, 2010
lec 2F.71