

In-Class Problems Week 2, Wed.

Problem 1.

Prove by truth table that OR distributes over AND:

$$[P \text{ OR } (Q \text{ AND } R)] \text{ is equivalent to } [(P \text{ OR } Q) \text{ AND } (P \text{ OR } R)] \quad (1)$$

Problem 2.

This problem¹ examines whether the following specifications are *satisfiable*:

1. If the file system is not locked, then
 - (a) new messages will be queued.
 - (b) new messages will be sent to the messages buffer.
 - (c) the system is functioning normally, and conversely, if the system is functioning normally, then the file system is not locked.
2. If new messages are not queued, then they will be sent to the messages buffer.
3. New messages will not be sent to the message buffer.

(a) Begin by translating the five specifications into propositional formulas using four propositional variables:

$L ::=$ file system locked,
 $Q ::=$ new messages are queued,
 $B ::=$ new messages are sent to the message buffer,
 $N ::=$ system functioning normally.

(b) Demonstrate that this set of specifications is satisfiable by describing a single truth assignment for the variables L, Q, B, N and verifying that under this assignment, all the specifications are true.

(c) Argue that the assignment determined in part (b) is the only one that does the job.

Problem 3.

When the Mathematician says to his student, “If a function is not continuous, then it is not differentiable,” then letting D stand for “differentiable” and C for continuous, the only proper translation of the Mathematician’s statement would be

$$\text{NOT}(C) \text{ IMPLIES } \text{NOT}(D),$$

or equivalently,

$$D \text{ IMPLIES } C.$$

But when a Mother says to her son, “If you don’t do your homework, then you can’t watch TV,” then letting T stand for “watch TV” and H for “do your homework,” a reasonable translation of the Mother’s statement would be

$$\text{NOT}(H) \text{ IFF } \text{NOT}(T),$$

or equivalently,

$$H \text{ IFF } T.$$

Explain why it is reasonable to translate these two IF-THEN statements in different ways into propositional formulas.

Problem 4.

Propositional logic comes up in digital circuit design using the convention that **T** corresponds to 1 and **F** to 0. A simple example is a 2-bit half-adder circuit. This circuit has 3 binary inputs, a_1, a_0 and b , and 3 binary outputs, c, o_1, o_0 . The 2-bit word a_1a_0 gives the binary representation of an integer, s between 0 and 3. The 3-bit word co_1o_0 gives the binary representation of $s + b$. The output c is called the *final carry bit*.

So if s and b were both 1, then the value of a_1a_0 would be 01 and the value of the output co_1o_0 would 010, namely, the 3-bit binary representation of $1 + 1$.

In fact, the final carry bit equals 1 only when all three binary inputs are 1, that is, when $s = 3$ and $b = 1$. In that case, the value of co_1o_0 is 100, namely, the binary representation of $3 + 1$.

This 2-bit half-adder could be described by the following formulas:

$$\begin{aligned} c_0 &= b \\ o_0 &= a_0 \text{ XOR } c_0 \\ c_1 &= a_0 \text{ AND } c_0 && \text{the carry into column 1} \\ o_1 &= a_1 \text{ XOR } c_1 \\ c_2 &= a_1 \text{ AND } c_1 && \text{the carry into column 2} \\ c &= c_2. \end{aligned}$$

(a) Generalize the above construction of a 2-bit half-adder to an $n + 1$ bit half-adder with inputs a_n, \dots, a_1, a_0 and b for arbitrary $n \geq 0$. That is, give simple formulas for o_i and c_i for $0 \leq i \leq n + 1$, where c_i is the carry into column i and $c = c_{n+1}$.

(b) Write similar definitions for the digits and carries in the sum of two $n + 1$ -bit binary numbers $a_n \dots a_1 a_0$ and $b_n \dots b_1 b_0$.

Visualized as digital circuits, the above adders consist of a sequence of single-digit half-adders or adders strung together in series. These circuits mimic ordinary pencil-and-paper addition, where a carry into a column is calculated directly from the carry into the previous column, and the carries have to ripple across all the columns before the carry into the final column is determined. Circuits with this design are called “ripple-carry” adders. Ripple-carry adders are easy to understand and remember and require a nearly minimal number of operations. But the higher-order output bits and the final carry take time proportional to n to reach their final values.

(c) How many of each of the propositional operations does your adder from part (b) use to calculate the sum?

The Propositional Operations

P	NOT P
T	F
F	T

P	Q	P AND Q
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	P OR Q
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	P XOR Q
T	T	F
T	F	T
F	T	T
F	F	F

P	Q	P IMPLIES Q
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	P IFF Q
T	T	T
T	F	F
F	T	F
F	F	T

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