Solutions to In-Class Problems Week 2, Mon.

Problem 1.
The proof below uses the Well Ordering Principle to prove that every amount of postage that can be paid exactly using only 6 cent and 15 cent stamps, is divisible by 3. Let the notation "$j \mid k$" indicate that integer j is a divisor of integer k, and let $S(n)$ mean that exactly n cents postage can be paid using only 6 and 15 cent stamps. Then the proof shows that

$$S(n) \text{ IMPLIES } 3 \mid n, \quad \text{for all nonnegative integers } n. \tag{*}$$

Fill in the missing portions (indicated by "...") of the following proof of (*).

Let C be the set of *counterexamples* to (*), namely[1]

$$C ::= \{n \mid \ldots\}$$

Solution. n is a counterexample to (*) if n cents postage can be made and n is not divisible by 3, so the predicate

$$S(n) \text{ and } \text{NOT}(3 \mid n)$$

defines the set, C, of counterexamples. ■

Assume for the purpose of obtaining a contradiction that C is nonempty. Then by the WOP, there is a smallest number, $m \in C$. This m must be positive because....

Solution. ...$3 \mid 0$, so 0 is not a counterexample. ■

But if $S(m)$ holds and m is positive, then $S(m-6)$ or $S(m-15)$ must hold, because....

Solution. ...if $m > 0$ cents postage is made from 6 and 15 cent stamps, at least one stamp must have been used, so removing this stamp will leave another amount of postage that can be made. ■

So suppose $S(m-6)$ holds. Then $3 \mid (m-6)$, because...

Solution. ...if $\text{NOT}(3 \mid (m-6))$, then $m-6$ would be a counterexample smaller than m, contradicting the minimality of m. ■

[1] The notation "$\{n \mid \ldots\}$" means "the set of elements, n, such that ...."

But if $3 \mid (m-6)$, then obviously $3 \mid m$, contradicting the fact that m is a counterexample.

Next suppose $S(m-15)$ holds. Then the proof for $m-6$ carries over directly for $m-15$ to yield a contradiction in this case as well. Since we get a contradiction in both cases, we conclude that...

Solution. ... C must be empty. That is, there are no counterexamples to (*), ■

which proves that (*) holds.

Problem 2.
Use the Well Ordering Principle to prove that

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \tag{1}$$

for all nonnegative integers, n.

Solution. The proof is by contradiction.

Suppose to the contrary that equation (1) failed for some $n \geq 0$. Then by the WOP, there is a *smallest* nonnegative integer, m, such that (1) does not hold when $n = m$.

But (1) clearly holds when $n = 0$, which means that $m \geq 1$. So $m-1$ is nonegative, and since it is smaller than m, equation (1) must be true for $n = m-1$. That is,

$$\sum_{k=0}^{m-1} k^2 = \frac{(m-1)((m-1)+1)(2(m-1)+1)}{6}. \tag{2}$$

Now add m^2 to both sides of equation (2). Then the left hand side equals

$$\sum_{k=0}^{m} k^2$$

and the right hand side equals

$$\frac{(m-1)((m-1)+1)(2(m-1)+1)}{6} + m^2$$

Now a little algebra (given below) shows that the right hand side equals

$$\frac{m(m+1)(2m+1)}{6}.$$

That is,

$$\sum_{k=0}^{m} k^2 = \frac{m(m+1)(2m+1)}{6},$$

contradicting the fact that equation (1) does not hold for m.

It follows that there is no smallest nonnegative integer for which equation (1) fails. Hence (1) must hold for all nonnegative integers.

Here's the algebra:

$$\begin{aligned}
\frac{(m-1)((m-1)+1)(2(m-1)+1)}{6} + m^2 &= \frac{(m-1)m(2m-1)}{6} + m^2 \\
&= \frac{(m^2-m)(2m-1)}{6} + m^2 \\
&= \frac{(2m^3-3m^2+m)}{6} + \frac{6m^2}{6} \\
&= \frac{(2m^3+3m^2+m)}{6} \\
&= \frac{m(m+1)(2m+1)}{6}
\end{aligned}$$

■

Problem 3.
Euler's Conjecture in 1769 was that there are no positive integer solutions to the equation

$$a^4 + b^4 + c^4 = d^4.$$

Integer values for a, b, c, d that do satisfy this equation, were first discovered in 1986. So Euler guessed wrong, but it took more two hundred years to prove it.

Now let's consider Lehman's[2] equation, similar to Euler's but with some coefficients:

$$8a^4 + 4b^4 + 2c^4 = d^4 \tag{3}$$

Prove that Lehman's equation (3) really does not have any positive integer solutions.

Hint: Consider the minimum value of a among all possible solutions to (3).

Solution. Suppose that there exists a solution. Then there must be a solution in which a has the smallest possible value. We will show that, in this solution, a, b, c, and d must all be even. However, we can then obtain another solution over the positive integers with a smaller a by dividing a, b, c, and d in half. This is a contradiction, and so no solution exists.

All that remains is to show that a, b, c, and d must all be even. The left side of Lehman's equation is even, so d^4 is even, so d must be even. Substituting $d = 2d'$ into Lehman's equation gives:

$$8a^4 + 4b^4 + 2c^4 = 16d'^4 \tag{4}$$

Now $2c^4$ must be a multiple of 4, since every other term is a multiple of 4. This implies that c^4 is even and so c is also even. Substituting $c = 2c'$ into the previous equation gives:

[2] Suggested by Eric Lehman, a former 6.042 Lecturer.

$$8a^4 + 4b^4 + 32c'^4 = 16d'^4 \tag{5}$$

Arguing in the same way, $4b^4$ must be a mutliple of 8, since every other term is. Therefore, b^4 is even and so b is even. Substituting $b = 2b'$ gives:

$$8a^4 + 64b'^4 + 32c'^4 = 16d'^4 \tag{6}$$

Finally, $8a^4$ must be a multiple of 16, a^4 must be even, and so a must also be even. Therefore, a, b, c, and d must all be even, as claimed. ■

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