In-Class Problems Week 2, Mon.

Problem 1.
The proof below uses the Well Ordering Principle to prove that every amount of postage that can be paid exactly using only 6 cent and 15 cent stamps, is divisible by 3. Let the notation "$j \mid k$" indicate that integer j is a divisor of integer k, and let $S(n)$ mean that exactly n cents postage can be paid using only 6 and 15 cent stamps. Then the proof shows that

$$S(n) \text{ IMPLIES } 3 \mid n, \quad \text{for all nonnegative integers } n. \qquad (*)$$

Fill in the missing portions (indicated by "...") of the following proof of (*).

> Let C be the set of *counterexamples* to (*), namely[1]
>
> $$C ::= \{n \mid \ldots\}$$
>
> Assume for the purpose of obtaining a contradiction that C is nonempty. Then by the WOP, there is a smallest number, $m \in C$. This m must be positive because....
>
> But if $S(m)$ holds and m is positive, then $S(m-6)$ or $S(m-15)$ must hold, because....
>
> So suppose $S(m-6)$ holds. Then $3 \mid (m-6)$, because...
>
> But if $3 \mid (m-6)$, then obviously $3 \mid m$, contradicting the fact that m is a counterexample.
>
> Next suppose $S(m-15)$ holds. Then the proof for $m-6$ carries over directly for $m-15$ to yield a contradiction in this case as well. Since we get a contradiction in both cases, we conclude that...
>
> which proves that (*) holds.

Problem 2.
Use the Well Ordering Principle to prove that

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \qquad (1)$$

for all nonnegative integers, n.

[1] The notation "$\{n \mid \ldots\}$" means "the set of elements, n, such that ...."

Problem 3.
Euler's Conjecture in 1769 was that there are no positive integer solutions to the equation

$$a^4 + b^4 + c^4 = d^4.$$

Integer values for a, b, c, d that do satisfy this equation, were first discovered in 1986. So Euler guessed wrong, but it took more two hundred years to prove it.

Now let's consider Lehman's[2] equation, similar to Euler's but with some coefficients:

$$8a^4 + 4b^4 + 2c^4 = d^4 \tag{2}$$

Prove that Lehman's equation (2) really does not have any positive integer solutions.

Hint: Consider the minimum value of a among all possible solutions to (2).

[2]Suggested by Eric Lehman, a former 6.042 Lecturer.

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