

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science
MIT 6.042J/18.062J

The Well Ordering Principle

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Albert R Meyer February, 8, 2010 Lec 2M.1

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Well Ordering principle

Every nonempty set of
nonnegative integers
has a
least element.

Familiar? Now you mention it, *Yes.*
Obvious? *Yes.*
Trivial? *Yes.* But *watch out:*

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6	9	13	7
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Well Ordering principle

Every nonempty set of
nonnegative rationals
has a
least element.

NO!

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Well Ordering principle

Every nonempty set of
~~*nonnegative integers*~~
has a
least element.

NO!

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Well Ordering Principle Proofs

To prove $\forall n \in \mathbb{N}. P(n)$ using WOP:

- define set of counterexamples
 $C ::= \{n \in \mathbb{N} \mid \text{NOT } P(n)\}$
- assume C is not empty. By WOP, have minimum element $m \in C$
- Reach a *contradiction* somehow ... usually by finding $c \in C$ with $c < m$

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Well Ordered Postage

available stamps:  

5¢ 3¢


Thm: Get any amount $n \geq 8¢$


Prove by WOP. Suppose *not*. Let m be *least* counterexample: if $m > n \geq 8$, can get $n¢$.


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Well Ordered Postage

$m > 8$: 

$m > 9$: 

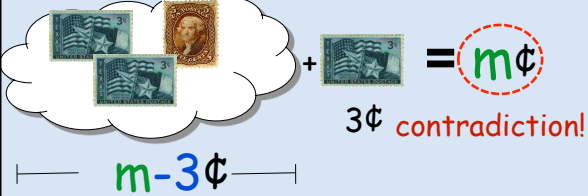
$m > 10$: 

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Well Ordered Postage

So $m \geq 11$. Now $m > m-3 \geq 8$
so can get $m-3\text{¢}$. But



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Geometric sums

$$1 + r + r^2 + r^3 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Proof by WOP. Let m be smallest n with \neq . But $=$ for $n = 0$, so $m > 0$, and

$$1 + r + r^2 + r^3 + \dots + r^{m-1} = \frac{r^m - 1}{r - 1}$$

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Geometric sums

$$1 + r + r^2 + r^3 + \dots + r^{m-1} = \frac{r^m - 1}{r - 1}$$

add r^m to both sides

$$\text{LHS} = 1 + r + r^2 + r^3 + \dots + r^{m-1} + r^m$$

$$\text{RHS} = \frac{r^m - 1}{r - 1} + \frac{r^{m+1} - r^m}{r - 1} = \frac{r^{m+1} - 1}{r - 1}$$

so $=$ at m , **contradicting \neq** :
there is no counterexample.

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Team Problems

Problems

1-3

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