Mathematics for Computer Science

MIT 6.042J/18.062J

The Well Ordering Principle

Well Ordering principle

Every nonempty set of *nonnegative integers* has a *least element.*

Familiar? Now you mention it, Yes.
Obvious? Yes.
Trivial? Yes. But watch out:

Well Ordering principle

Every nonempty set of *nonnegative* rationals has a *least element.*

NO!

Well Ordering principle

Every nonempty set of ~~nonnegative~~ *integers* has a *least element.*

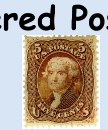
NO!

Well Ordering Principle Proofs

To prove $\forall n \in \mathbb{N}. P(n)$ using WOP:

- define set of counterexamples $C ::= \{n \in \mathbb{N} \mid \text{NOT } P(n)\}$
- assume C is not empty. By WOP, have minimum element $m \in C$
- Reach a contradiction *somehow* … usually by finding $c \in C$ with $c < m$

Well Ordered Postage

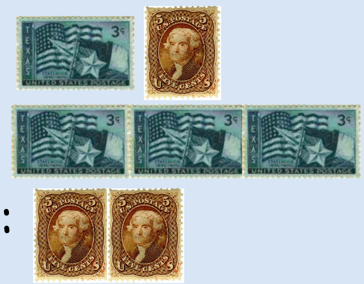
available stamps:  

5¢ 3¢

Thm: Get any amount $n \geq 8$¢

Prove by WOP. Suppose not. Let m be least counterexample: if $m > n \geq 8$, can get n¢.

Well Ordered Postage



$m > 8$:

$m > 9$:

$m > 10$:

Well Ordered Postage

So $m \geq 11$. Now $m > m\text{-}3 \geq 8$ so can get $m\text{-}3$¢. But

+ 3¢ = m¢ contradiction!

m-3¢

Geometric sums

$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

Proof by WOP. Let m be smallest n with $\neq$. But $=$ for $n = 0$, so $m > 0$, and

$$1 + r + r^2 + r^3 + \cdots + r^{m-1} = \frac{r^m - 1}{r - 1}$$

Geometric sums

$$1 + r + r^2 + r^3 + \cdots + r^{m-1} = \frac{r^m - 1}{r - 1}$$

add r^m to both sides

$$\text{LHS} = 1 + r + r^2 + r^3 + \cdots + r^{m-1} + r^m$$

$$\text{RHS} = \frac{r^m - 1}{r - 1} + \frac{r^{m+1} - r^m}{r - 1} = \frac{r^{m+1} - 1}{r - 1}$$

so $=$ at m, contradicting $\neq$: there is no counterexample.

Team Problems

Problems 1–3