

In-Class Problems Week 1, Fri.

Problem 1.

Generalize the proof from lecture (reproduced below) that $\sqrt{2}$ is irrational, for example, how about $\sqrt[3]{2}$? Remember that an irrational number is a number that cannot be expressed as a ratio of two integers.

Theorem. $\sqrt{2}$ is an irrational number.

Proof. The proof is by contradiction: assume that $\sqrt{2}$ is rational, that is,

$$\sqrt{2} = \frac{n}{d}, \tag{1}$$

where n and d are integers. Now consider the smallest such positive integer denominator, d . We will prove in a moment that the numerator, n , and the denominator, d , are both even. This implies that

$$\frac{n/2}{d/2}$$

is a fraction equal to $\sqrt{2}$ with a smaller positive integer denominator, a contradiction.

Since the assumption that $\sqrt{2}$ is rational leads to this contradiction, the assumption must be false. That is, $\sqrt{2}$ is indeed irrational. This italicized comment on the implication of the contradiction normally goes without saying, but since this is the first 6.042 exercise about proof by contradiction, we've said it.

To prove that n and d have 2 as a common factor, we start by squaring both sides of (1) and get $2 = n^2/d^2$, so

$$2d^2 = n^2. \tag{2}$$

So 2 is a factor of n^2 , which is only possible if 2 is in fact a factor of n .

This means that $n = 2k$ for some integer, k , so

$$n^2 = (2k)^2 = 4k^2. \tag{3}$$

Combining (2) and (3) gives $2d^2 = 4k^2$, so

$$d^2 = 2k^2. \tag{4}$$

So 2 is a factor of d^2 , which again is only possible if 2 is in fact also a factor of d , as claimed. ■

Problem 2.

Here is a generalization of Problem 1 that you may not have thought of:

Lemma 2.1. *Let the coefficients of the polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + x^n$ be integers. Then any real root of the polynomial is either integral or irrational.*

(a) Explain why Lemma 2.1 immediately implies that $\sqrt[m]{k}$ is irrational whenever k is not an m th power of some integer.

(b) Collaborate with your tablemates to write a clear, textbook quality proof of Lemma 2.1 on your whiteboard. (Besides clarity and correctness, textbook quality requires good English with proper punctuation. When a real textbook writer does this, it usually takes multiple revisions; if you're satisfied with your first draft, you're probably misjudging.) You may find it helpful to appeal to the following:

Lemma 2.2. *If a prime, p , is a factor of some power of an integer, then it is a factor of that integer.*

You may assume Lemma 2.2 without writing down its proof, but see if you can explain why it is true.

Problem 3.

If we raise an irrational number to an irrational power, can the result be rational? Show that it can by considering $\sqrt{2}^{\sqrt{2}}$ and arguing by cases.

Problem 4.

Here is a different proof that $\sqrt{2}$ is irrational, taken from the American Mathematical Monthly, v.116, #1, Jan. 2009, p.69:

Proof. Suppose for the sake of contradiction that $\sqrt{2}$ is rational, and choose the least integer, $q > 0$, such that $(\sqrt{2} - 1)q$ is a nonnegative integer. Let $q' ::= (\sqrt{2} - 1)q$. Clearly $0 < q' < q$. But an easy computation shows that $(\sqrt{2} - 1)q'$ is a nonnegative integer, contradicting the minimality of q . ■

(a) This proof was written for an audience of college teachers, and is a little more concise than desirable at this point in 6.042. Write out a more complete version which includes an explanation of each step.

(b) Now that you have justified the steps in this proof, do you have a preference for one of these proofs over the other? Why? Discuss these questions with your teammates for a few minutes and summarize your team's answers on your whiteboard.

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