Solutions to Mini-Quiz May 5

Problem 1 (**6 points**).
You would like to give a bouquet for Mother's Day. You find an online service that will make bouquets of **lilies**, **roses** and **tulips**, subject to the following constraints:

- there must be at most 3 lilies,
- there can be any number of roses,
- there must be a multiple of four tulips.

Example: A bouquet of 4 tulips, 5 roses and no lilies satisfies the constraints.

Let f_n be the number of possible bouquets with n flowers that fit the service's constraints. Express $F(x)$, the generating function corresponding to $\langle f_0, f_1, f_2, \dots \rangle$, as a quotient of polynomials (or products of polynomials). You do not need to simplify this expression.

Solution. Generating function for the number of ways to choose lilies:

$$F_L(x) = 1 + x + x^2 + x^3 = \frac{1 - x^4}{1 - x}$$

Generating function for the number of ways to choose roses:

$$F_R(x) = 1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1 - x}$$

Generating function for the number of ways to choose tulips:

$$F_W(x) = 1 + x^3 + x^7 + \cdots = \frac{1}{1 - x^4}$$

By the Convolution Property, the generating function for f_n is therefore the product of these functions, namely,

$$\begin{aligned} F(x) &= F_L(x) F_R(x) F_W(x) \\ &= \frac{(1 + x + x^2 + x^3)}{(1 - x)(1 - x^2)} \\ &= \frac{(1 - x^4)}{(1 - x)^2 (1 - x^4)} \\ &= \frac{1}{(1 - x)^2}. \end{aligned}$$

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Problem 2 (**8 points**).
[A Baseball Series]
The New York Yankees and the Boston Red Sox are playing a two-out-of-three series. (In other words, they play until one team has won two games. Then that team is declared the overall winner and the series ends.) Assume that the Red Sox win each game with probability $2/3$, regardless of the outcomes of previous games.

Answer the questions below using the four step method. You can use the same tree diagram for all three problems.

(a) What is the probability that only 2 games are played?

Solution. This means that either the Yankees win both games, which occurs with probability $(1/3)^2$ or the Red Sox win both games, which occurs with probability $(2/3)^2$.

Summing these yields $(1/3)^2 + (2/3)^2 = 5/9$

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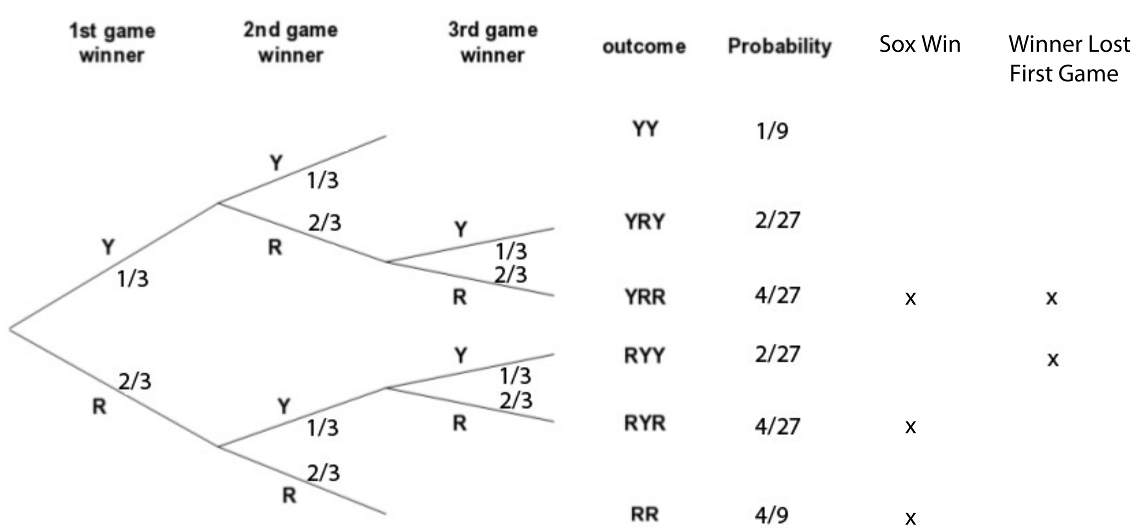
(b) What is the probability that the winner of the series loses the first game?

Solution. We use the four step method with the graph below. ■

(c) What is the probability that the Red Sox loses the series?

Solution. A tree diagram is worked out below.



From the tree diagram, we get:

$$\Pr\{\text{Winner Lost First Game}\} = \frac{4}{27} + \frac{2}{27} = \frac{6}{27}$$
$$\Pr\{\text{Sox Lost}\} = \frac{3}{27} + \frac{2}{27} + \frac{2}{27} = \frac{7}{27}$$

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Problem 3 (6 points).
The following combinational identity is known as Pascal's Triangle:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

(a) Give a *combinatorial proof* that for Pascal's Triangle.

Solution. Both sides are equal to:

$$\frac{n!}{k!\,(n-k)!}$$

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(b) Verify this combinational proof by giving an algebraic proof of this same fact.

Solution. This is identical to problem 16.12.1 in the notes, reproduced below:

Jay figures that n people (including himself) are competing for spots on a team and only k will be selected. As part of maneuvering for a spot on the team, he needs to work out how many different teams are possible. There are two cases to consider:

- Jay *is* selected for the team, and his $k-1$ teammates are selected from among the other $n-1$ competitors. The number of different teams that can be formed in this way is:

$$\binom{n-1}{k-1}$$

- Jay is *not* selected for the team, and all k team members are selected from among the other $n-1$ competitors. The number of teams that can be formed this way is:

$$\binom{n-1}{k}$$

All teams of the first type contain Jay, and no team of the second type does; therefore, the two sets of teams are disjoint. Thus, by the Sum Rule, the total number of teams is:

$$\binom{n-1}{k-1} + \binom{n-1}{k}$$

Another way of thinking about it is that n people (including himself) are trying out for k spots. Thus, the number of ways to select the team is simply:

$$\binom{n}{k}$$

Each method correctly counts the number of teams, so the answers must be equal. So we know:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

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