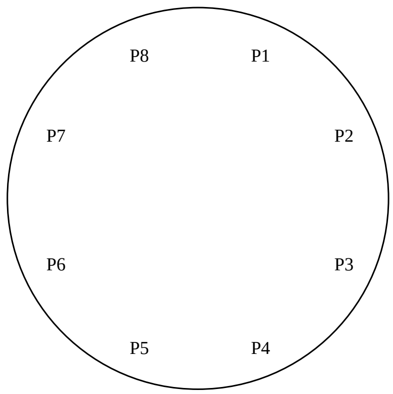
Solutions to Mini-Quiz Apr. 21

Problem 1 (13 points).
The queen and king of hearts decide to host a poker game and invite their six fellow royals —the queen and king of clubs, diamonds, and spades. The queen of hearts has a round table with eight chairs $P_1, P_2, \ldots, P_8$ for these eight people:



You may answer each of the following questions with a numerical expression that uses factorials and arithmetic operations.

(a) In how many ways can the queen assign the eight people to different chairs?

Solution. $8!$ ■

A *seating* is a circular arrangement of people around the table in which all that matters is who sits next to whom, not which chairs they are in. In other words, two ways of assigning people to chairs define the *same seating* when one assignment is a rotation of the other. For example, the following two assignments of people to chairs define the same seating:

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$K\spadesuit$	$Q\spadesuit$	$K\heartsuit$	$Q\heartsuit$	$K\diamondsuit$	$Q\diamondsuit$	$K\clubsuit$	$Q\clubsuit$
$K\diamondsuit$	$Q\diamondsuit$	$K\clubsuit$	$Q\clubsuit$	$K\spadesuit$	$Q\spadesuit$	$K\heartsuit$	$Q\heartsuit$

(b) How many different *seatings* are there?

Solution.

$$\frac{8!}{8} = 7!$$

■

(c) How many distinct *seatings* are there if the queen and king of hearts must be seated next to each other? *Hint:* Think of the queen and king as one unit, but remember the king and queen can be in either order.

Solution.

$$2! \cdot \frac{7!}{7} = 2 \cdot 6!$$

■

(d) How many distinct *seatings* are there if the queen and king of hearts must be seated next to each other, and the queen and king of spades must also be seated next to each other?

Solution.

$$2! \cdot 2! \cdot \frac{6!}{6} = 4 \cdot 5!$$

■

(e) How many distinct *seatings* are there where no one is seated next to their spouse?

Hint: Let N_1 be the answer to part (c), N_2 the answer to part (d), N_3 be the number of seatings with each of the $\spadesuit$, $\heartsuit$, and $\diamondsuit$ couples seated next to their spouses, and N_4 be the number of seatings with everyone next to their spouse. Use inclusion-exclusion and express your answer in terms of the N_i's.

Solution.

$$\begin{aligned}
\text{Answer} &= \{\text{Total seatings}\} \\
&- \binom{4}{1} \times \{\text{Seatings with at least one couple seated together}\} \\
&+ \binom{4}{2} \times \{\text{Seatings with at least two couples seated together}\} \\
&- \binom{4}{3} \times \{\text{Seatings with at least three couples seated together}\} \\
&+ \binom{4}{4} \times \{\text{Seatings with the four couples seated together}\} \\
&= 7! - 4 \cdot (2 \cdot 6!) + 6 \cdot (4 \cdot 5!) - 4 \cdot (8 \cdot 4!) + 1 \cdot (16 \cdot 3!)
\end{aligned}$$

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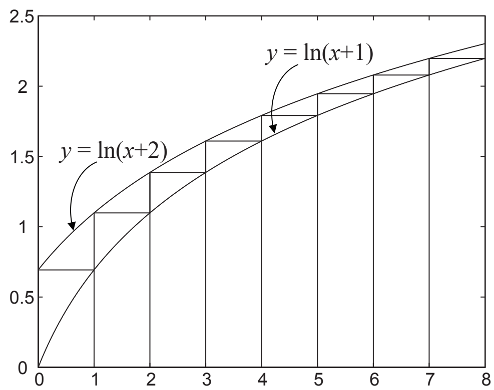
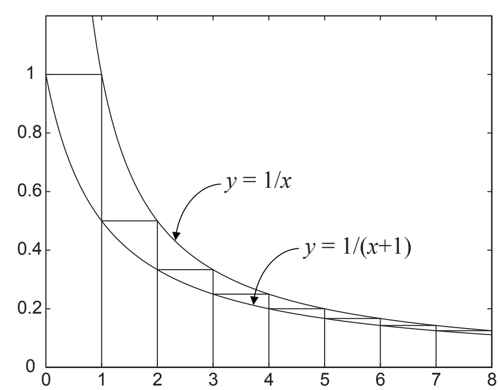
Problem 2 (3 points).
Assume n is an integer larger than 1. Circle all the correct inequalities below.

Explanations are not required, but partial credit for wrong answers will not be given without them. *Hint:* You may find the graphs helpful.





- $\displaystyle\sum_{i=1}^{n} \ln(i+1) \leq \ln 2 + \int_1^n \ln(x+1)dx$
- $\displaystyle\sum_{i=1}^{n} \ln(i+1) \leq \int_0^n \ln(x+2)dx$
- $\displaystyle\sum_{i=1}^{n} \frac{1}{i} \geq \int_0^n \frac{1}{x+1}dx$

Solution. The 2nd and 3rd inequalities hold. ■

Problem 3 (4 points).
Circle each of the true statements below.

Explanations are not required, but partial credit for wrong answers will not be given without them.

- $n^2 \sim n^2 + n$
- $3^n = O\left(2^n\right)$
- $n^{\sin(n\pi/2)+1} = o\left(n^2\right)$
- $n = \Theta\left(\dfrac{3n^3}{(n+1)(n-1)}\right)$

Solution. The 1st and 4th statements are true. ■