Solutions to Mini-Quiz Mar. 17

Problem 1 (**8 points**).
Starting with some number of 4-cent and 7-cent stamps on the table, there are two ways to change the stamps:

(i) Add *one* 4-cent stamp, or

(ii) remove *two* 4-cent AND *two* 7-cent stamps (when this is possible).

(a) Let A be the number of 4-cent stamps; and B be the number of 7-cent stamps. The chart below indicates properties of some derived variables; fill it in.

derived variables:	B	$4A + 7B$	$\text{rem}(B, 2)$	$\text{rem}(4A + 7B, 2)$
weakly increasing				
strictly increasing				
weakly decreasing				
strictly increasing				
constant				

Solution.

derived variables:	B	$4A + 7B$	$\text{rem}(B, 2)$	$\text{rem}(4A + 7B, 2)$
weakly increasing	NO	NO	YES	YES
strictly increasing	NO	NO	NO	NO
weakly decreasing	YES	NO	YES	YES
strictly increasing	NO	NO	NO	NO
constant	NO	NO	YES	YES

■

(b) Circle the properties below that are preserved invariants:

1. The number of 7-cent stamps (B) must be even.
2. The number of 7-cent stamps (B) must be greater than 0.
3. The total postage ($4A + 7B$) on the table must be odd.
4. $4A > 7B$.

Solution. (1), (3), (4). ■

(c) Using the Invariant Principle, show that it is impossible to have stamps with a total value of exactly 90 cents on the table when we start with exactly 211 7-cent stamps. (You may use without proof the preserved invariance of some of the properties from part (b).)

Solution. We will show that the predicate (3) must hold for all reachable states of the state machine.

First, we check that the predicate holds for the start state:

$$\text{rem}(211 \cdot 7 + 0 \cdot 4, 2) = \text{rem}(1477, 2) = 1$$

So the total cost of stamps was clearly odd in the start state.

Since (3) is a preserved invariant that holds for the start state, it must hold for all reachable states of the machine.

However, since the predicate does *not* hold for the state of having *exactly 90 cents*, it is not a reachable state and it is therefore impossible to have exactly 90 cents on the table.

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Problem 2 (**6 points**).
Covering edges were introduced in class problem: if a and b are distinct vertices of a digraph, then a is said to *cover* b if there is an edge from a to b and every path from a to b traverses this edge. If a covers b, the edge from a to b is called a *covering edge*.

Let D be a finite directed acyclic graph (DAG).

(a) If there is a path in D from a vertex, u, to vertex, v, explain why there must be a *longest* path from u to v.

Solution. If D has m vertices, then no path can be longer $m - 1$ —otherwise some vertex must repeat on the path, which means there would be a cycle, contradicting the fact that D is a DAG. So there must be a *longest* path from u to v. (Technically, this follows from the Well Ordering Principle applied to the set $\{v - n \in \mathbb{N} \mid \text{there is a path of length } n \text{ from } u \text{ to } v\}$.) ■

(b) Give a proof of the following claim from the class problem:
Claim. *If there is a path in D from a vertex, u, to vertex, v, then there is a path from u to v that only traverses covering edges.*

Solution. By part (a), there is a longest path from u to v. If some edge on this path was not a covering edge, then by definition there is a path of length 2 or more between its endpoints, and replacing this edge by the path would yield a longer path from u to v, a contradiction. Hence all edges must be covering edges. ■

(c) Show that the Claim fails for the finite digraph, F, with three vertices and edges from every vertex to every other vertex. *Hint:* What are the covering edges of F?

Solution. There are no covering edges in F, since for each edge $u \to v$ there is a length 2 path uwv through the remaining vertex, w, that does not traverse this edge. So there is no path of covering edges from any vertex to any other vertex. ■

Problem 3 (6 points).
Let G be a connected simple graph. Prove that if an edge in a connected graph is not traversed by any simple cycle, then it is a cut edge.[1]

Solution. *Proof.* Supppose edge u—v is not a cut-edge. We show that it must be traversed by a simple cycle.

Since the edge is not a cut-edge, the graph obtained by removing the edge is connected. So there exists a path from u to v which does not traverse u—v. We proved in lecture that the shortest such path from u to v must be simple. But this simple path together with u—v is a simple cycle that traverses u—v.

■

[1] A simple cycle is a subgraph of G isomorphic to the cycle graph C_n for $n \geq 3$. An edge is a *cut-edge* when removing the edge disconnects the graph.

6.042J / 18.062J Mathematics for Computer Science
Spring 2010