Solutions to Mini-Quiz Mar. 3

Problem 1 (**6 points**).
Each year, Santa's reindeer hold "Reindeer Games," from which Rudolph is pointedly excluded. The Games consist of a sequence of matches between pairs of reindeer. Draws are not possible.

On Christmas Eve, Santa produces a rank list of all his reindeer. If reindeer p lost a match to reindeer q, then p appears below q in Santa's ranking, but if he has any choice because of unplayed matches, Santa can give higher rank to the reindeer he likes better. To prevent confusion, two reindeer may not play a match if either outcome would lead to a cycle of reindeer, where each lost to the next.

Though it is only March, the 2010 Reindeer Games have already begun (punctuality is key at the north pole). We can describe the results so far with a binary relation, L, on the set of reindeer, where pLq means that reindeer p lost in a match to reindeer q. Let $\prec$ be the corresponding *indirectly lost* relation, where reindeer p indirectly lost to reindeer q when p lost a match with q, or when p lost to a reindeer who lost to q, or when p lost to a reindeer who lost to a reindeer who lost to q, etc. Note that the "indirectly lost" relation, $\prec$, is a partial order.

On the following page you'll find a list of terms and a sequence of statements. Add the appropriate term to each statement.

Terms

a strict partial order	a weak partial order	a total order
comparable elements	incomparable elements	a chain
an antichain	a maximal antichain	a topological sort
a minimum element	a minimal element	
a maximum element	a maximal element	

Statements

(a) A reindeer who is unbeaten so far is

________________________________ of the partial order $\prec$.

Solution. a maximal element ■

(b) A reindeer who has lost every match so far is

________________________________ of the partial order $\prec$.

Solution. a minimal element ■

(c) Two reindeer can *not* play a match if they are

________________________________ of $\prec$.

Solution. comparable elements ■

(d) A reindeer assured of first place in Santa's ranking is

________________________________ of $\prec$.

Solution. a maximum element ■

(e) A sequence of reindeer which *must* appear in the same order in Santa's rank list is

________________________________.

Solution. a chain ■

(f) A set of reindeer such that any two could still play a match is

________________________________.

Solution. an antichain ■

(g) The fact that no reindeer loses a match to himself implies that $\prec$ is

________________________________.

Solution. a strict partial order ■

(h) Santa's final ranking of his reindeer on Christmas Eve must be

____ of $\prec$.

Solution. a topological sort ■

(i) No more matches are possible if and only if $\prec$ is

____.

Solution. a total order ■

Problem 2 (7 points).
Prove *by induction* that every exact amount of postage of 12 cents or more can be formed using 3 and 7 cent stamps. In particular, clearly identify

- the induction variable,
- the induction hypothesis,
- the base case(s), and
- the inductive step.

Solution. *Proof.* The following proof is by strong induction on n with induction hypothesis

$$S(n) ::= \text{exactly } n \text{ cents postage can be formed using 3 and 7 cent stamps.}$$

Base cases: $S(12)$, $S(13)$ and $S(14)$ are shown to hold by explicit calculations:

$$\begin{aligned} 12 &= 3+3+3+3, \\ 13 &= 3+3+7, \\ 14 &= 7+7. \end{aligned}$$

Inductive step: By strong induction, we may assume that $S(k)$ holds for $12 \leq k \leq n$ and must then prove that $S(n+1)$ is true.

Now if $n+1 \leq 14$, then $S(n+1)$ follows from the base case. On the other hand, if $n+1 > 14$, then $n-2 \geq 12$, so $S(n-2)$ is true by induction hypothesis. So by adding one 3 cent stamp to the stamps that form $n-2$ cents postage, we will have postage equal to $n-2+3 = n+1$ cents, showing that $S(n+1)$ is true.

It follows by strong induction that $P(n)$ holds for all $n \geq 14$. ■

Problem 3 (7 points).
Let $[\mathbb{N} \to \{1,2,3\}]$ be the set of infinite sequences containing only the numbers 1, 2, and 3. For example, some sequences of this kind are:

$$(1,1,1,1...),$$
$$(2,2,2,2...),$$
$$(3,2,1,3...).$$

Prove that $[\mathbb{N} \to \{1,2,3\}]$ is uncountable.

Hint: One approach is to define a surjective function from $[\mathbb{N} \to \{1,2,3\}]$ to the power set $\mathcal{P}(\mathbb{N})$.

Solution. *Proof.* We can define a surjective function from $f : [\mathbb{N} \to \{1,2,3\}] \to \mathcal{P}(\mathbb{N})$ as follows:

$$f(s) ::= \{n \in \mathbb{N} \mid s[n] = 1\}$$

where $s[n]$ is the nth element of sequence s.

Now if there was a surjective function from $g : \mathbb{N} \to [\mathbb{N} \to \{1,2,3\}]$, then the composition of f and g would be a surjective function from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$ contradicting Theorem 5.2.1 in the text. ■

Proof. Alternatively, to show that $[\mathbb{N} \to \{1,2,3\}]$ is uncountable, we show that no function, $\sigma : \mathbb{N} \to [\mathbb{N} \to \{1,2,3\}]$ is a surjection. In particular, we will describe a sequence diag $\in [\mathbb{N} \to \{1,2,3\}]$ such that diag $\notin$ range (σ).

Let

$$\sigma_0, \sigma_1, \ldots$$

be the sequences in the range of σ. Then we can define diag as follows:

$$\text{diag} ::= r(\sigma_0[0]), r(\sigma_1[1]), r(\sigma_2[2]), \ldots,$$

where $r : \{1,2,3\} \to \{1,2,3\}$ is some function such that $r(i) \neq i$ for $i = 1,2,3$.

Now by definition,

$$\text{diag}[n] \neq \sigma_n[n],$$

for all $n \in \mathbb{N}$, proving that diag is not in the range of σ, as claimed. ■

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