Solutions to Mini-Quiz Feb. 17

Problem 1 (**5 points**).
Prove that $\log_4 9$ is irrational. Your proof should be clear and well-organized, and should explicitly indicate where particular properties of primes are assumed.

Solution. *Proof.* Suppose to the contrary that $\log_4 9 = m/n$ for some integers m and n. Since $\log_4 9$ is positive, we may assume that m and n are also positive. So we have

$$\begin{aligned} \log_4 9 &= m/n \\ 4^{\log_4 9} &= 4^{m/n} \\ 9 &= (4^m)^{1/n} \\ 9^n &= 4^m \end{aligned} \tag{1}$$

But this is impossible, since left hand side of (1) is odd, but the right hand side is even.

This contradiction implies that $\log_4 9$ must be irrational. ■

Problem 2 (**10 points**).
Let A be the set of five propositional formulas shown below on the left, and let C be the set of three propositional formulas on the right. Let R be the "implies" binary relation from A to C which is defined by the rule

$$F \; R \; G \quad \text{iff} \quad [\text{the formula } (F \text{ IMPLIES } G) \text{ is valid}].$$

For example, $(P \text{ AND } Q) \; R \; P$, because the formula $(P \text{ AND } Q)$ does imply P. Also, it is not true that $(P \text{ OR } Q) \; R \; P$ since $[(P \text{ OR } Q)$ does not imply P.

(a) Fill in the arrows so the following figure describes the graph of the relation, R:

A	arrows	C
M		
		Q
P OR Q		
		$\overline{P}$ OR $\overline{Q}$
P XOR Q		
		M AND (P IMPLIES M)
P AND Q		
NOT(P AND Q)		

Solution. Four arrows for R:

M	iff	M AND (P IMPLIES M)
P XOR Q	implies	$\overline{P}$ OR $\overline{Q}$
P AND Q	implies	Q
NOT(P AND Q)	iff	$\overline{P}$ OR $\overline{Q}$

■

(b) Circle the properties below possessed by the relation R:

FUNCTION TOTAL INJECTIVE SURJECTIVE BIJECTIVE

Solution. From part (a), the "implies" relation, R, is a surjective function. ■

(c) Circle the properties below possessed by the relation R^{-1}:

FUNCTION TOTAL INJECTIVE SURJECTIVE BIJECTIVE

Solution. From part (b), the inverse relation, R^{-1}, is a total injection. ■

Problem 3 (5 points).
Prove by the Well Ordering Principle that for all nonnegative integers, n:

$$\sum_{i=0}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 .$$

The proof is by contradiction.

Suppose to the contrary that this failed for some $n \geq 0$. Then by the WOP, there is a *smallest* nonnegative integer, m, such this formula does not hold when $n = m$.

But it clearly holds when $n = 0$, which means that $m \geq 1$. So $m - 1$ is nonegative, and since it is smaller than m, the formula must be true for $n = m - 1$. That is,

$$\sum_{i=0}^{m-1} i^3 = \left(\frac{(m-1)m}{2}\right)^2. \tag{2}$$

Now add m^3 to both sides of equation (2). Then the left hand side equals

$$\sum_{i=0}^{m} i^3$$

and the right hand side equals

$$\left(\frac{(m-1)m}{2}\right)^2 + m^3$$

Now a little algebra shows that the right hand side equals

$$\left(\frac{m(m+1)}{2}\right)^2.$$

That is,

$$\sum_{i=0}^{m} i^3 = \left(\frac{m(m+1)}{2}\right)^2,$$

contradicting the fact that our formula does not hold for m.

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