Final Examination

Your name: ______________________________

- This exam is **closed book** except for a three page, 2-sided crib sheet. Total time is 3 hours.
- Write your solutions in the space provided with your name on every page. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem's page.
- GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

Problem	Points	Grade	Grader
1	10		
2	10		
3	10		
4	10		
5	10		
6	10		
7	10		
8	10		
9	10		
10	10		
Total	100		

Problem 1 (Probable Satisfiability) (10 points).
A *literal* is a propositional variable or its negation. A *k-clause* is an OR of k literals, with no variable occurring more than once in the clause. For example,

$$P \text{ OR } \overline{Q} \text{ OR } \overline{R} \text{ OR } V,$$

is a 4-clause, but

$$\overline{V} \text{ OR } \overline{Q} \text{ OR } \overline{X} \text{ OR } V,$$

is not, since V appears twice.

Let $\mathcal{S}$ be a set of n distinct k-clauses involving v variables. The variables in different k-clauses may overlap or be completely different, so $k \le v \le nk$.

A random assignment of true/false values will be made independently to each of the v variables, with true and false assignments equally likely. Write formulas in n, k, and v in answer to the first two parts below.

(a) (2 points) What is the probability that the last k-clause in $\mathcal{S}$ is true under the random assignment?

(b) (3 points) What is the expected number of true k-clauses in $\mathcal{S}$?

(c) (5 points) A set of propositions is *satisfiable* iff there is an assignment to the variables that makes all of the propositions true. Use your answer to part (b) to prove that if $n < 2^k$, then $\mathcal{S}$ is satisfiable.

Problem 2 (**Asymptotic Bounds and Partial Orders**) (**10 points**).

For each of the relations below, indicate whether it is *transitive* but not a partial order (**Tr**), a *total order* (**Tot**), a *strict partial order* that is not total (**S**), a *weak partial order* that is not total (**W**), or *none* of the above (**N**).

- the "is a subgraph of" relation on graphs. _______
  (Note that every graph is considered a subgraph of itself.)

Let f, g be nonnegative functions on the real numbers.

- the "Big Oh" relation, $f = O(g)$, _______
- the "Little Oh" relation, $f = o(g)$, _______
- the "asymptotically equal" relation, $f \sim g$. _______

Problem 3 (Graph Coloring & Induction) (10 points).
Recall that a *coloring* of a graph is an assignment of a color to each vertex such that no two adjacent vertices have the same color. A *k-coloring* is a coloring that uses at most k colors.

False Claim. *Let G be a graph whose vertex degrees are all $\leq k$. If G has a vertex of degree strictly less than k, then G is k-colorable.*

(a) (2 points) Give a counterexample to the False Claim when $k = 2$.

(b) (4 points) Underline the exact sentence or part of a sentence that is the first unjustified step in the following "proof" of the False Claim.

False proof. Proof by induction on the number n of vertices:

Induction hypothesis:

$P(n)$::= "Let G be an n-vertex graph whose vertex degrees are all $\leq k$. If G also has a vertex of degree strictly less than k, then G is k-colorable."

Base case: ($n = 1$) G has one vertex, the degree of which is 0. Since G is 1-colorable, $P(1)$ holds.

Inductive step:

We may assume $P(n)$. To prove $P(n+1)$, let G_{n+1} be a graph with $n+1$ vertices whose vertex degrees are all k or less. Also, suppose G_{n+1} has a vertex, v, of degree strictly less than k. Now we only need to prove that G_{n+1} is k-colorable.

To do this, first remove the vertex v to produce a graph, G_n, with n vertices. Let u be a vertex that is adjacent to v in G_{n+1}. Removing v reduces the degree of u by 1. So in G_n, vertex u has degree strictly less than k. Since no edges were added, the vertex degrees of G_n remain $\leq k$. So G_n satisfies the conditions of the induction hypothesis, $P(n)$, and so we conclude that G_n is k-colorable.

Now a k-coloring of G_n gives a coloring of all the vertices of G_{n+1}, except for v. Since v has degree less than k, there will be fewer than k colors assigned to the nodes adjacent to v. So among the k possible colors, there will be a color not used to color these adjacent nodes, and this color can be assigned to v to form a k-coloring of G_{n+1}. ■

(c) (4 points) With a slightly strengthened condition, the preceding proof of the False Claim could be revised into a sound proof of the following Claim:

Claim. *Let G be a graph whose vertex degrees are all $\leq k$. If* $\langle$statement inserted from below$\rangle$ *has a vertex of degree strictly less than k, then G is k-colorable.*

Circle each of the statements below that could be inserted to make the Claim true.

- G is connected and
- G has no vertex of degree zero and
- G does not contain a complete graph on k vertices and
- every connected component of G
- some connected component of G

Problem 4 (Planar Embeddings) (10 points).
The planar graph embeddings in class (repeated in the Appendix) were only defined for connected planar graphs. The definition can be extended to planar graphs that are not necessarily connected by adding the following additional constructor case to the definition:

- **Constructor Case:** (collect disjoint graphs) Suppose $\mathcal{E}$ and $\mathcal{F}$ are planar embeddings with no vertices in common. Then $\mathcal{E} \cup \mathcal{F}$ is a planar embedding.

Euler's Planar Graph Theorem now generalizes to unconnected graphs as follows: if a planar embedding, $\mathcal{E}$, has v vertices, e edges, f faces, and c connected components, then

$$v - e + f - 2c = 0. \tag{1}$$

This can be proved by structural induction on the definition of planar embedding.

(a) (4 points) State and prove the base case of the structural induction.

(b) (2 points) Carefully state what must be proved in the new constructor case (collect disjoint graphs) of the structural induction.

(c) (4 points) Prove the (collect disjoint graphs) case of the structural induction.

Problem 5 (Euler's Function) (10 points).

(a) (2 points) What is the value of $\phi(175)$, where ϕ is Euler's function?

(b) (3 points) Call a number from 0 to 174 *powerful* iff some positive power of the number is congruent to 1 modulo 175. What is the probability that a random number from 0 to 174 is powerful?

(c) (5 points) What is the remainder of $(-12)^{482}$ divided by 175?

Problem 6 (**Magic Trick Redux**) (**10 points**).
In this problem we consider the famous 6.042 magic trick. Unlike the one performed in class by the TAs, this time the Assistant will be choosing 4 cards and revealing 3 of them to the Magician (in some particular order) instead of choosing 5 and revealing 4.

(a) Show that the Magician could not pull off this trick with a deck larger than 27 cards.

(b) Show that, in principle, the Magician could pull off the Card Trick with a deck of exactly 27 cards. (You do not need to describe the actual method.)

Problem 7 (Combinatorial Proof) (10 points).

(a) (2 points) Let S be a set with i elements. How many ways are there to divide S into a pair of subsets?

(b) (4 points)

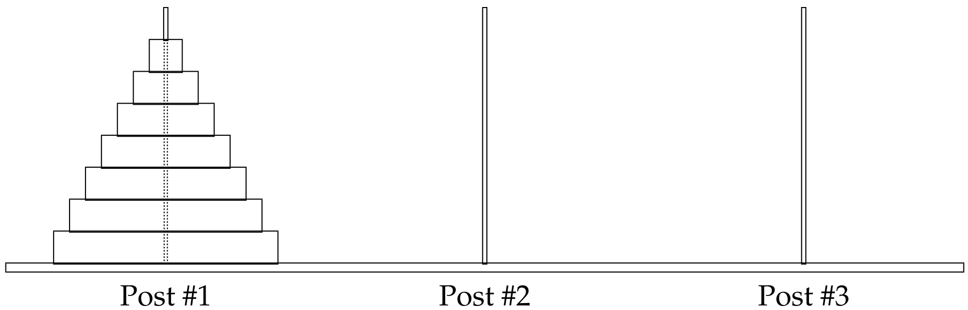
Here is a combinatorial proof of an equation giving a closed form for a certain summation $\sum_{i=0}^{n}$:

> There are n marbles, each of which is to be painted red, green, blue, or yellow. One way to assign colors is to choose red, green, blue, or yellow successively for each marble.
>
> An alternative way to assign colors to the marbles is to
>
> - choose a number, i, between 0 and n,
> - choose a set, S, of i marbles,
> - divide S into two subsets; paint the first subset red and the other subset green.
> - divide the set of all the marbles not in S into two subsets; paint the first subset blue and the other subset yellow.

What is the equation?

(c) (4 points) Now use the binomial theorem to prove the same equation.

Problem 8 (Linear Recurrence) (10 points).
Less well-known than the Towers of Hanoi —but no less fascinating —are the Towers of Sheboygan. As in Hanoi, the puzzle in Sheboygan involves 3 posts and n disks of different sizes. Initially, all the disks are on post #1:



The objective is to transfer all n disks to post #2 via a sequence of moves. A move consists of removing the top disk from one post and dropping it onto another post with the restriction that a larger disk can never lie above a smaller disk. Furthermore, a local ordinance requires that *a disk can be moved only from a post to the next post on its right —or from post #3 to post #1.* Thus, for example, moving a disk directly from post #1 to post #3 is not permitted.

(a) (2 points) One procedure that solves the Sheboygan puzzle is defined recursively: to move an initial stack of n disks to the next post, move the top stack of $n-1$ disks to the furthest post by moving it to the next post two times, then move the big, nth disk to the next post, and finally move the top stack another two times to land on top of the big disk. Let s_n be the number of moves that this procedure uses. Write a simple linear recurrence for s_n.

(b) (4 points) Let $S(x)$ be the generating function for the sequence $\langle s_0, s_1, s_2, \ldots \rangle$. Carefully show that

$$S(x) = \frac{x}{(1-x)(1-4x)}.$$

(c) (4 points) Give a simple formula for s_n.

Problem 9 (Variance & Deviation) (10 points).
The hat-check staff has had a long day serving at a party, and at the end of the party they simply return people's hats at random. Assume that n people checked hats at the party.

Let X_i be the indicator variable for the ith person getting their own hat back. Let S_n be the total number of people who get their own hat back.

(a) (1 point) What is the expected number of people who get their own hat back?

(b) (2 points) Write a simple formula for $\mathrm{E}[X_i X_j]$ for $i \neq j$. *Hint:* What is $\Pr\{X_j = 1 \mid X_i = 1\}$?

(c) (3 points) Show that $\mathrm{E}\left[S_n^2\right] = 2$. *Hint:* $X_i^2 = X_i$.

(d) (1 point) What is the variance of S_n?

(e) (3 points) Use the Chebyshev bound to show that the probability that 11 or more people get their own hat back is at most 0.01.

Problem 10 (Sampling & Confidence) (10 points).
Yesterday, the bakers at a local cake factory baked a huge number of cakes. To estimate the fraction, b, of cakes in this program that are improperly prefared, the cake-testers will take a small sample of cakes chosen randomly and independently (so it is possible, though unlikely, that the same cake might be chosen more than once). For each cake chosen, they perform a variety of non-destructive tests to determine if the cake is improperly prepared, after which they will use the fraction of bad cakes in their sample as their estimate of the fraction b.

The factory statistician can use estimates of a binomial distribution to calculate a value, s, for a number of cakes to sample which ensures that with 97% confidence, the fraction of bad cakes in the sample will be within 0.006 of the actual fraction, b, of bad cakes in the back.

Mathematically, the *batch* is an actual outcome that already happened. The *sample* is a random variable defined by the process for randomly choosing s cakes from the batch. The justification for the statistician's confidence depends on some properties of the batch and how the sample of s cakes from the batch are chosen. These properties are described in some of the statements below. Mark each of these statements as **T** (true) or **F** (false), and then briefly explain your answer.

1. The probability that the ninth cake in the *batch* is bad is b. ________

2. All cakes in the batch are equally likely to be the third cake chosen in the *sample*. ________

3. The probability that the ninth cake chosen for the *sample* is bad, is b. ________

4. Given that the first cake chosen for the *sample* is bad, the probability that the second cake chosen will also be bad is greater than b.

   ____

5. Given that the last cake in the *batch* is bad, the probability that the next-to-last cake in the batch will also be bad is greater than b.

   ____

6. Given that the first two cakes selected in the *sample* are the same kind of cake —they might both be chocoloate, or both be angel food cakes,. . . —the probability that the first cake is bad may be greater than b.

   ____

7. The expectation of the indicator variable for the last cake in the *sample* being bad is b._______

8. There is zero probability that all the cakes in the *sample* will be different. _______