Problem Set 10

Due: April 23

Reading: Notes Ch. 16.10–16.12

Problem 1.
Let's develop a proof of the Inclusion-Exclusion formula using high school algebra.

(a) Most high school students will get freaked by the following formula, even though they actually know the rule it expresses. How would you explain it to them?

$$\prod_{i=1}^{n}(1-x_i) = \sum_{I\subseteq\{1,\dots,n\}} (-1)^{|I|} \prod_{j\in I} x_j. \tag{1}$$

Hint: Show them an example.

For any set, S, let M_S be the *membership* function of S:

$$M_S(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } x \notin S. \end{cases}$$

Let $S_1, \dots, S_n$ be a sequence of finite sets, and abbreviate M_{S_i} as M_i. Let the domain of discourse, D, be the union of the S_i's. That is, we let

$$D ::= \bigcup_{i=1}^{n} S_i,$$

and take complements with respect to D, that is,

$$\overline{T} ::= D - T,$$

for $T \subseteq D$.

(b) Verify that for $T \subseteq D$ and $I \subseteq \{1, \dots n\}$,

$$M_{\overline{T}} = 1 - M_T, \tag{2}$$

$$M_{(\bigcap_{i\in I} S_i)} = \prod_{i\in I} M_{S_i}, \tag{3}$$

$$M_{(\bigcup_{i\in I} S_i)} = 1 - \prod_{i\in I}(1 - M_i). \tag{4}$$

(Note that (3) holds when I is empty because, by convention, an empty product equals 1, and an empty intersection equals the domain of discourse, D.)

(c) Use (1) and (4) to prove

$$M_D = \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \prod_{j \in I} M_j. \tag{5}$$

(d) Prove that

$$|T| = \sum_{u \in D} M_T(u). \tag{6}$$

(e) Now use the previous parts to prove

$$|D| = \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \left| \bigcap_{i \in I} S_i \right| \tag{7}$$

(f) Finally, explain why (7) immediately implies the usual form of the Inclusion-Exclusion Principle:

$$|D| = \sum_{i=1}^{n} (-1)^{i+1} \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I| = i}} \left| \bigcap_{j \in I} S_j \right|. \tag{8}$$

Problem 2.
Give combinatorial proofs of the identities below. Use the following structure for each proof. First, define an appropriate set S. Next, show that the left side of the equation counts the number of elements in S. Then show that, from another perspective, the right side of the equation also counts the number of elements in set S. Conclude that the left side must be equal to the right, since both are equal to $|S|$.

(a)

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k} \cdot \binom{n}{n-k}$$

(b)

$$\sum_{i=0}^{r} \binom{n+i}{i} = \binom{n+r+1}{r}$$

Hint: consider a set of binary strings that could be counted using the right side of the equation, then try partitioning them into subsets countable by the elements of the sum on the left.

Problem 3.
According to the Multinomial Theorem 16.11.3, $(x_1 + x_2 + \ldots + x_k)^n$ can be expressed as a sum of terms of the form

$$\binom{n}{r_1, r_2, \ldots, r_k} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}.$$

(a) How many terms are there in the sum?

(b) The sum of these multinomial coefficients has an easily expressed value:

$$\sum_{\substack{r_1+r_2+\ldots+r_k=n,\\ r_i\in\mathbb{N}}} \binom{n}{r_1,\ r_2,\ \ldots,\ r_k} = k^n \tag{9}$$

Give a combinatorial proof of this identity.

Hint: How many terms are there when $(x_1 + x_2 + \ldots + x_k)^n$ is expressed as a sum of monomials in x_i *before* terms with like powers of these variables are collected together under a single coefficient?

Student's Solutions to Problem Set 10

Your name:	
Due date:	April 23
Submission date:	
Circle your TA/LA:	Megumi Tom Richard Eli

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
Total	

[1]People other than course staff.
[2]Give citations to texts and material other than the Spring '10 course materials.