

## Problem Set 8

Due: April 9

**Reading:** Notes Ch. 14; Ch. 15

### Problem 1.

Suppose  $m, n$  are relatively prime. In the problem you will prove the key property of Euler's function that  $\phi(mn) = \phi(m)\phi(n)$ .

(a) Prove that for any  $a, b$ , there is an  $x$  such that

$$x \equiv a \pmod{m}, \tag{1}$$

$$x \equiv b \pmod{n}. \tag{2}$$

*Hint:* Congruence (1) holds iff

$$x = jm + a. \tag{3}$$

for some  $j$ . So there is such an  $x$  only if

$$jm + a \equiv b \pmod{n}. \tag{4}$$

Solve (4) for  $j$ .

(b) Prove that there is an  $x$  satisfying the congruences (1) and (2) such that  $0 \leq x < mn$ .

(c) Prove that the  $x$  satisfying part (b) is unique.

(d) For an integer  $k$ , let  $k^*$  be the integers between 1 and  $k - 1$  that are relatively prime to  $k$ . Conclude from part (c) that the function

$$f : (mn)^* \rightarrow m^* \times n^*$$

defined by

$$f(x) ::= (\text{rem}(x, m), \text{rem}(x, n))$$

is a bijection.

(e) Conclude from the preceding parts of this problem that

$$\phi(mn) = \phi(m)\phi(n).$$

**Problem 2 (4).**

Use integration to find upper and lower bounds that differ by at most 0.1 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

$$\sum_{i=1}^{\infty} \frac{1}{(2i+1)^2}$$

**Problem 3.**

Indicate which of the following holds for each pair of functions  $(f(n), g(n))$  in the table below. Assume  $k \geq 1$ ,  $\epsilon > 0$ , and  $c > 1$  are constants. Pick the four table entries you consider to be the most challenging or interesting and justify your answers to these.

$f(n)$	$g(n)$	$f = O(g)$	$f = o(g)$	$g = O(f)$	$g = o(f)$	$f = \Theta(g)$	$f \sim g$
$2^n$	$2^{n/2}$						
$\sqrt{n}$	$n^{\sin n\pi/2}$						
$\log(n!)$	$\log(n^n)$						
$n^k$	$c^n$						
$\log^k n$	$n^\epsilon$						

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## Student's Solutions to Problem Set 8

<b>Your name:</b>				
<b>Due date:</b>	April 9			
<b>Submission date:</b>				
<b>Circle your TA/LA:</b>	Megumi	Tom	Richard	Eli

**Collaboration statement:** Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:  
got help from:<sup>1</sup>  
and referred to:<sup>2</sup>

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**DO NOT WRITE BELOW THIS LINE**

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Problem	Score
1	
2	
3	
Total	

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Spring 2010

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