Problem Set 7

Due: April 2

Reading: Notes Ch. 12; Ch. 14

Problem 1.
A simple graph is *sevenish* when it has no simple cycles of length less than seven. A *map* is a planar graph whose faces are all simple cycles (no dongles or bridges). Let M be a sevenish, connected, map with $v > 2$ vertices, e edges and f faces.

(a) Prove that

$$e \leq (7v - 14)/5. \tag{1}$$

Hint: Similar to the proof of Corollary 12.6.3, that $e \leq 3v - 6$ in planar graphs.

(b) Show that M has a vertex of degree at most two.

(c) Part (b) can be used to prove that M is 3-colorable by induction on v. The proof is slightly complicated by the fact that subgraphs of M may not be sevenish, connected, maps. For each of the following properties, briefly explain why all connected subgraphs of M have the property, or give an example of a connected subgraph of an M that does not have the property.

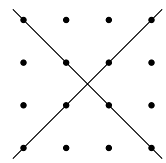
1. map
2. planar
3. sevenish
4. connected
5. 3-colorable

Problem 2.
Prove that the greatest common divisor of three integers a, b, and c is equal to their smallest positive linear combination; that is, the smallest positive value of $sa + tb + uc$, where s, t, and u are integers.

Problem 3.
Two nonparallel lines in the real plane intersect at a point. Algebraically, this means that the equations

$$y = m_1 x + b_1$$
$$y = m_2 x + b_2$$

have a unique solution (x, y), provided $m_1 \neq m_2$. This statement would be false if we restricted x and y to the integers, since the two lines could cross at a noninteger point:



However, an analogous statement holds if we work over the integers *modulo a prime*, p. Find a solution to the congruences

$$y \equiv m_1 x + b_1 \pmod{p}$$
$$y \equiv m_2 x + b_2 \pmod{p}$$

when $m_1 \not\equiv m_2 \pmod{p}$. Express your solution in the form $x \equiv ? \pmod{p}$ and $y \equiv ? \pmod{p}$ where the ?'s denote expressions involving m_1, m_2, b_1, and b_2. You may find it helpful to solve the original equations over the reals first.

Problem 4.
Let $S_k = 1^k + 2^k + \ldots + (p-1)^k$, where p is an odd prime and k is a positive multiple of $p-1$. Use Fermat's theorem to prove that $S_k \equiv -1 \pmod{p}$.

Student's Solutions to Problem Set 7

Your name:	
Due date:	April 2
Submission date:	
Circle your TA/LA:	Megumi Tom Richard Eli

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
Total	

[1]People other than course staff.
[2]Give citations to texts and material other than the Spring '10 course materials.