Problem Set 6

Due: March 19

Reading: Notes Ch. 10.3– 10.6; Ch. 11

Problem 1.
An edge is said to *leave* a set of vertices if one end of the edge is in the set and the other end is not.

(a) An n-node graph is said to be *mangled* if there is an edge leaving every set of $\lfloor n/2 \rfloor$ or fewer vertices. Prove the following claim.
Claim. *Every mangled graph is connected.*

An n-node graph is said to be *tangled* if there is an edge leaving every set of $\lceil n/3 \rceil$ or fewer vertices.

(b) Draw a tangled graph that is not connected.

(c) Find the error in the proof of the following
False Claim. *Every tangled graph is connected.*

False proof. The proof is by strong induction on the number of vertices in the graph. Let $P(n)$ be the proposition that if an n-node graph is tangled, then it is connected. In the base case, $P(1)$ is true because the graph consisting of a single node is trivially connected.

For the inductive case, assume $n \geq 1$ and $P(1), \ldots, P(n)$ hold. We must prove $P(n+1)$, namely, that if an $(n+1)$-node graph is tangled, then it is connected.

So let G be a tangled, $(n+1)$-node graph. Choose $\lceil n/3 \rceil$ of the vertices and let G_1 be the tangled subgraph of G with these vertices and G_2 be the tangled subgraph with the rest of the vertices. Note that since $n \geq 1$, the graph G has a least two vertices, and so both G_1 and G_2 contain at least one vertex. Since G_1 and G_2 are tangled, we may assume by strong induction that both are connected. Also, since G is tangled, there is an edge leaving the vertices of G_1 which necessarily connects to a vertex of G_2. This means there is a path between any two vertices of G: a path within one subgraph if both vertices are in the same subgraph, and a path traversing the connecting edge if the vertices are in separate subgraphs. Therefore, the entire graph, G, is connected. This completes the proof of the inductive case, and the Claim follows by strong induction. ■

Problem 2.
My computer program has seven variables, `t`, `u`, `v`, `w`, `x`, `y`, `z`, and computes in 6 steps. The steps in which each variable is used are as follows: `t:` steps 1 through 6; `u:` step 2; `v:` steps 2 through

4; `w:` steps 1,3 and 5; `x:` steps 1 and 6; `y:` steps 3 through 6; `z:` steps 4 and 5. Each variable will need to occupy the same index register at each one of the steps in which it is going to be used; however during the steps when the variable is not being used the register may be used by some other variable. How many such registers are needed for my program? Explain how you arrived at the answer.

Problem 3.
The set Supersymm of "super-symmetric strings" is defined recursively as follows:

Base Case: The 26 lower case letters of the Roman alphabet, `a`, `b`,..., `z`, are in Supersymm.

Constructor Case: If α and β are strings in Supersymm, then the string $\alpha\beta\alpha$ is in Supersymm.

(a) Which of the following are super-symmetric strings? Briefly explain your answers.

(i) `a`

(ii) `aaaba`

(iii) `abcbacabcba`

(iv) λ, the empty string

(v) `abaabcbaaba`

(b) Prove by structural induction that in any super-symmetric string, exactly one letter appears an odd number of times.

Problem 4.
Take a regular deck of 52 cards. Each card has a suit and a value. The suit is one of four possibilities: heart, diamond, club, spade. The value is one of 13 possibilities, $A, 2, 3, \ldots, 10, J, Q, K$. There is exactly one card for each of the 4×13 possible combinations of suit and value.

Ask your friend to lay the cards out into a grid with 4 rows and 13 columns. They can fill the cards in any way they'd like. In this problem you will show that you can always pick out 13 cards, one from each column of the grid, so that you wind up with cards of all 13 possible values.

(a) Explain how to model this trick as a bipartite matching problem between the 13 column vertices and the 13 value vertices. Is the graph necessarily degree constrained?

(b) Show that any n columns must contain at least n different values and prove that a matching must exist.

Student's Solutions to Problem Set 6

Your name:

Due date: March 19

Submission date:

Circle your TA/LA: Megumi Tom Richard Eli

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
Total	

[1]People other than course staff.
[2]Give citations to texts and material other than the Spring '10 course materials.