Problem Set 5

Due: March 12

Reading: Ch. 9.1.4, Derived Variables; Ch. 9.2, Stable Marriage; Ch. 10.1, Graph Isomorphism

Problem 1.
The following procedure can be applied to any digraph, G:

1. Delete an edge that is traversed by a directed cycle.
2. Delete edge $u \to v$ if there is a directed path from vertex u to vertex v that does not traverse $u \to v$.
3. Add edge $u \to v$ if there is no directed path in either direction between vertex u and vertex v.

Repeat these operations until none of them are applicable.

This procedure can be modeled as a state machine. The start state is G, and the states are all possible digraphs with the same vertices as G.

(a) Let G be the graph with vertices $\{1, 2, 3, 4\}$ and edges

$$\{1 \to 2, 2 \to 3, 3 \to 4, 3 \to 2, 1 \to 4\}$$

What are the possible final states reachable from G?

A *line graph* is a graph whose edges can all be traversed by a directed simple path. All the final graphs in part (a) are line graphs.

(b) Prove that if the procedure terminates with a digraph, H, then H is a line graph with the same vertices as G.

Hint: Show that if H is *not* a line graph, then some operation must be applicable.

(c) Prove that being a DAG is a preserved invariant of the procedure.

(d) Prove that if G is a DAG and the procedure terminates, then the path relation of the final line graph is a topological sort of G.

Hint: Verify that the predicate

$$P(u, v) ::= \text{there is a directed path from } u \text{ to } v$$

is a preserved invariant of the procedure, for any two vertices u, v of a DAG.

(e) Prove that if G is finite, then the procedure terminates.

Hint: Let s be the number of simple cycles, e be the number of edges, and p be the number of pairs of vertices with a directed path (in either direction) between them. Note that $p \leq n^2$ where n is the number of vertices of G. Find coefficients a, b, c such that $as + bp + e + c$ is a strictly decreasing, $\mathbb{N}$-valued variable.

Problem 2.
Give an example of a stable matching between 3 boys and 3 girls where no person gets their first choice. Briefly explain why your matching is stable.

Problem 3.
In a stable matching between n boys and girls produced by the Mating Ritual, call a person *lucky* if they are matched up with one of their $\lceil n/2 \rceil$ top choices. We will prove:

Theorem. *There must be at least one lucky person.*

To prove this, define the following derived variables for the Mating Ritual:

$q(B) = j$, where j is the rank of the girl that boy B is courting. That is to say, boy B is always courting the jth girl on his list.

$r(G)$ is the number of boys that girl G has rejected.

(a) Let

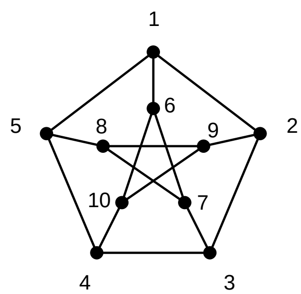
$$S ::= \sum_{B \in \text{Boys}} q(B) - \sum_{G \in \text{Girls}} r(G). \tag{1}$$

Show that S remains the same from one day to the next in the Mating Ritual.

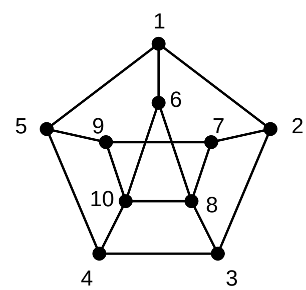
(b) Prove the Theorem above. (You may assume for simplicity that n is even.)

Hint: A girl is sure to be lucky if she has rejected half the boys.

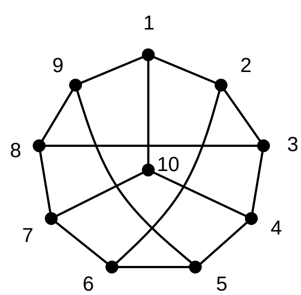
Problem 4.
Determine which among the four graphs pictured in the Figures are isomorphic. If two of these graphs are isomorphic, describe an isomorphism between them. If they are not, give a property that is preserved under isomorphism such that one graph has the property, but the other does not. For at least one of the properties you choose, *prove* that it is indeed preserved under isomorphism (you only need prove one of them).



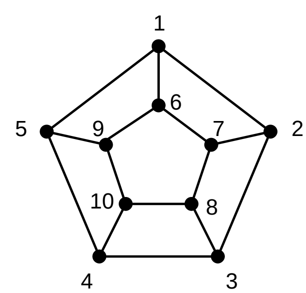
(a) G_1



(b) G_2



(c) G_3



(d) G_4

Figure 1: Which graphs are isomorphic?

Student's Solutions to Problem Set 5

Your name:	
Due date:	March 12
Submission date:	
Circle your TA/LA:	Megumi Tom Richard Eli

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
Total	

[1]People other than course staff.

[2]Give citations to texts and material other than the Spring '10 course materials.