Problem Set 4

Due: March 5

Reading: Chapter 7, *Partial Orders*, §§4–6; Ch. 8, Digraphs; Ch. 9, State Machines, §9.1–§9.1.3

Problem 1.
Let $\prec$ be a partial order on a set, A, and let

$$A_k ::= \{a \mid \text{depth}(a) = k\}$$

where $k \in \mathbb{N}$.

(a) Prove that $A_0, A_1, \ldots$ is a parallel schedule for $\prec$ according to Definition 7.5.5.

(b) Prove that A_k is an antichain.

Problem 2.
In a *round-robin* tournament, every pair of distinct players play against each other just once. For a round-robin tournament with with no tied games, a record of who beat whom can be described with a *tournament digraph*, where the vertices correspond to players and there is an edge $x \rightarrow y$ if x beat y in their game.

A *ranking* is a directed simple path that includes all the players.

(a) Give an example of a tournament digraph with more than one ranking.

(b) If a tournament digraph is a DAG, then it has a unique ranking. Explain.

(c) Prove that every tournament digraph has a ranking. *Hint:* Induction on the size of the tournament.

Problem 3.
A robot named Wall-E wanders around a two-dimensional grid. He starts out at $(0, 0)$ and is allowed to take four different types of step:

1. $(+2, -1)$
2. $(+1, -2)$
3. $(+1, +1)$

4. $(-3, 0)$

Thus, for example, Wall-E might walk as follows. The types of his steps are listed above the arrows.

$$(0,0) \xrightarrow{1} (2,-1) \xrightarrow{3} (3,0) \xrightarrow{2} (4,-2) \xrightarrow{4} (1,-2) \to \dots$$

Wall-E's true love, the fashionable and high-powered robot, Eve, awaits at $(0, 2)$.

(a) Describe a state machine model of this problem.

(b) Will Wall-E ever find his true love? Either find a path from Wall-E to Eve or use the Invariant Principle to prove that no such path exists.

Problem 4.
In the late 1960s, the military junta that ousted the government of the small republic of Nerdia completely outlawed built-in multiplication operations, and also forbade division by any number other than 3. Fortunately, a young dissident found a way to help the population multiply any two nonnegative integers without risking persecution by the junta. The procedure he taught people is:

```
procedure multiply(x, y: nonnegative integers)
r := x;
s := y;
a := 0;
while s ≠ 0 do
    if 3 | s then
        r := r + r + r;
        s := s/3;
    else if 3 | (s − 1) then
        a := a + r;
        r := r + r + r;
        s := (s − 1)/3;
    else
        a := a + r + r;
        r := r + r + r;
        s := (s − 2)/3;
return a;
```

We can model the algorithm as a state machine whose states are triples of nonnegative integers (r, s, a). The initial state is $(x, y, 0)$. The transitions are given by the rule that for $s > 0$:

$$(r, s, a) \to \begin{cases} (3r, s/3, a) & \text{if } 3 \mid s \\ (3r, (s-1)/3, a+r) & \text{if } 3 \mid (s-1) \\ (3r, (s-2)/3, a+2r) & \text{otherwise.} \end{cases}$$

(a) List the sequence of steps that appears in the execution of the algorithm for inputs $x = 5$ and $y = 10$.

(b) Use the Invariant Method to prove that the algorithm is partially correct —that is, if $s = 0$, then $a = xy$.

(c) Prove that the algorithm terminates after at most $1 + \log_3 y$ executions of the body of the `do` statement.

Student's Solutions to Problem Set 4

Your name:	
Due date:	March 5
Submission date:	
Circle your TA/LA:	Megumi Tom Richard Eli

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
Total	

[1]People other than course staff.
[2]Give citations to texts and material other than the Spring '10 course materials.