Problem Set 2

Due: February 19

Reading: Chapter 4, *Mathematical Data Types*; Chapter 5, *First-Order Logic*. (Assigned readings do not include the Problem sections.)

Reminder: Comments on the reading using the *NB online annotation system* are due at times indicated in the online tutor problem set TP.3. Reading Comments count for 5% of the final grade.

Problem 1.
Let $f : A \mapsto B$ and $g : B \mapsto C$ be functions.

(a) Prove that if the composition $g \circ f$ is a bijection, then f is an injection and g is a surjection.

(b) If f is an injection and g is a surjection, then is $g \circ f$ necessarily a bijection?

Problem 2.
There is a simple and useful way to extend composition of functions to composition of relations. Namely, let $R : B \rightarrow C$ and $S : A \rightarrow B$ be relations. Then the composition of R with S is the binary relation $(R \circ S) : A \rightarrow C$ defined by the rule

$$a \ (R \circ S) \ c ::= \ \exists b \in B. \, (b \ R \ c) \text{ AND } (a \ S \ b).$$

This agrees with the Definition 4.3.1 of composition in the special case when R and S are functions.

We can represent a relation, S, between two sets $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$ as an $n \times m$ matrix, M_S, of zeroes and ones, with the elements of M_S defined by the rule

$$M_S(i, j) = 1 \quad \text{IFF} \quad a_i \ S \ b_j.$$

If we represent relations as matrices in this fashion, then we can compute the composition of two relations R and S by a "boolean" matrix multiplication, $\otimes$, of their matrices. Boolean matrix multiplication is the same as matrix multiplication except that addition is replaced by OR and multiplication is replaced by AND. Namely, suppose $R : B \rightarrow C$ is a binary relation with $C = \{c_1, \ldots, c_p\}$. So M_R is an $m \times p$ matrix. Then $M_S \otimes M_R$ is an $n \times p$ matrix defined by the rule:

$$[M_S \otimes M_R](i, j) ::= \text{OR}_{k=1}^{m}[M_S(i, k) \text{ AND } M_R(k, j)]. \tag{1}$$

Prove that the matrix representation, $M_{R \circ S}$, of $R \circ S$ equals $M_S \otimes M_R$ (note the reversal of R and S).

Problem 3.
The standard notation for the proposition that a set, x, is a member of a set, y, is

$$x \in y.$$

Formulas built up from membership formulas of this form using logical connectives and quantifiers are called the logical *formulas of set theory*. For example, to say $x \neq y$ with a formula of set theory, we could write

$$\exists z\ (z \in x) \text{ XOR } (z \in y). \tag{2}$$

(a) Write a formula of set theory that means that x is the empty set.

(b) Write a formula of set theory that means that $x \subseteq y$, that is, x is a subset of y.

(c) Write a formula of set theory that means that $x \subset y$, that is, x is a *proper* subset of y.

(d) Write a formula of set theory that means that y is the powerset of x.

(e) Write a formula, Three(y), of set theory that means that y has at least three elements.

(f) Write a formula of set theory that means that y has *exactly* two elements.

(g) A set, x, is *member-minimal* in a set y iff x is a member of y and no element of y is a member of x. Write a formula, MM(x, y), of set theory that means that x is member-minimal in y.

(h) The *Foundation Axiom* of Zermelo-Fraenkel set theory asserts that every nonempty set has a member-minimal element. Express the Foundation Axiom as a formula of set theory. (You may use the formula MM(x, y) of part (g) as an abbreviation in your formula).

Problem 4.
Prove that for any sets A, B, C, and D, if $A \times B$ and $C \times D$ are disjoint, then either A and C are disjoint or B and D are disjoint.

Student's Solutions to Problem Set 2

Your name:	
Due date:	February 19
Submission date:	
Circle your TA/LA:	Megumi Tom Richard Eli

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	
2	
3	
4	
Total	

[1] People other than course staff.

[2] Give citations to texts and material other than the Spring '10 course materials.